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THE
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THE
ASTROPHYSICAL JOURNAL
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VOLUME LV JANUARY 1922 NUMBER 1

COLOR-TRANSPARENCY OF RAY-FILTERS IN USE AT YERKES OBSERVATORY

By ERNEST C. BRYANT

ABSTRACT

Spectro-photometric transmission curves for two ray-filters in use at Yerkes Observatory, and for three colored glasses used in the Hess-Ives tint photometer are given for 400 to 750 μμ. The filters investigated are J 1, M 1, December 1907, Beta 7 and Beta 10; the glasses are a red, a green and a blue-violet.

Lemon-Brace spectro-photometer.—Suggestions as to the adjustment and calibration of this instrument, which was used in obtaining the foregoing curves, are given, and also measurements of the *relative selective absorption of nicol prism and silver strip* from 475 to 750 μμ.

Color-curve of the 40-inch objective at Yerkes Observatory is reproduced in Figure 5.

The 40-inch refractor at Yerkes Observatory, although designed only for visual observations, has been available for astronomical photography since 1900 by means of a method perfected by G. W. Ritchey and described by him in the *Astrophysical Journal* for December of that year. The method consists of the use of a ray-filter which absorbs those waves of short lengths for which the objective is uncorrected, combined with photographic plates which are especially sensitive to the yellow, a color which the ray-filter transmits freely, and which comes in the region for which the color-curve of the 40-inch objective is very nearly flat.

The filters first used by Ritchey were made by Carbutt, of Philadelphia. R. J. Wallace came to Yerkes in 1906 and immediately began fitting filters to the various telescopes, especially to give photo-visual results in connection with the isochromatic plates in use, mainly the Cramer Isochromatic and Trichromatic.

After several trials the 8×10 filter marked "Dec. '07," consisting of a colored gelatine film between two sheets of optical glass, was put into regular use in the parallax program and other work with the 40-inch telescope. After some years the gelatine began to deteriorate at the edges of the film, and the filter was finally superseded by a similar one made by Mees, of the Eastman Research Laboratory. He planned to duplicate the former filter as closely as possible. Of the two which he made, that called "M 1" was put into use in the parallax work in May 1920.

Wallace made similar filters for the 2-foot reflector and the 6-inch ultra-violet camera. On the reflector, filter "Beta 7" 3×3 inches in size, was used from July 1906 until it was accidentally broken in July 1917. On the camera, "Beta 10," 4×5 inches, has been in use from February 1907 till the present time. Both have been shown to give substantially correct photo-visual magnitudes when used with Cramer Isochromatic plates. By comparison with photographic magnitudes found with ordinary plates on the same instruments, color-indices are obtained and spectral types inferred (see *Astrophysical Journal*, **27**, 169, April 1908).

After the accidental breaking of "Beta 7," several others were tried, both Wallace and Eastman; but none was satisfactory. In 1917 Petitdidier furnished two tinted Jena glass filters, No. F 4351, one 3×3 inches which replaced "Beta 7"; and one 6×6 inches. The former, called "J 1," has been used on the reflector since 1917.

Photographic determinations of the spectral intensity-curves for Seed 27 plates without ray-filter and for Cramer Trichromatic plates with ray-filter have been made, but no direct measurements of the color transmission of the filters themselves have previously been obtained. The loan of a Lemon-Brace Polarization Spectrophotometer from the Ryerson Physical Laboratory has enabled these determinations to be made this summer, and the following results have been obtained. This instrument has been described in this *Journal*, January 1900, September 1902, and April 1914, but a brief statement of the method used by the author for adjusting and calibrating it may be helpful to those having their first experience with the process.

Referring to Figure 1, the instrument when in adjustment must fulfil the following conditions: (1) slits S_1, S_2, and S_3 must be at the principal foci of their respective objectives; (2) optical axes of collimators C_1 and C_2 and of the telescope T must be in the same plane, which must be perpendicular to the refracting edges of the Brace prism AEI; (3) light from S_1 must strike prism at B, making $AB = 1/4AE$, at the angle of minimum deviation for the Fraunhofer lines D, emerging at F, making $IF = 1/4IE$; (4) light from S_2 similarly must strike the prism at D, making $ED = 1/4EA$, at the angle of minimum deviation for the D lines. It must meet the division plane CI at H, the same place where the light from S_1 meets it, and that part reflected by the silver strip on CI must follow the same direction HF, FJ as the light from S_1. In the instrument used, the Brace prism merely rested on its platform with nothing to locate its position on the platform or its orientation.

After making adjustments (1) and (2), beams of sunlight were reflected into S_1 and S_2, directed so as to emerge through the centers of their objectives. This latter condition was carefully maintained throughout the process. The prism was oriented for minimum deviation of D lines, the slits were opened and coincidence of emerging beams at F was tested by placing a piece of white paper against the face of the prism at that point. The prism was moved parallel to line S_1B, retaining minimum deviation of D lines, until the beam from S_2 reflected at H emerged at F exactly between the beams from S_1 which passed above and below the silver strip at H. Coincidence in direction of emergent beams was tested by receiving them on the white paper held in front of the telescope objective and then beyond the eyepiece J. If not coincident in direction, collimator C_2 must be swung on its axis to obtain coincidence. This will require a repetition of the preceding adjustment. These two adjustments must be thus repeated until the three beams emerge at F in the same vertical line, enter the telescope and emerge at the eyepiece likewise in the same vertical line. The final adjustment of C_2 is made by narrowing S_1 and S_2 and testing for coincidence of Fraunhofer lines. The lines given by light through S_2 should be exact continuations of the lines given by light through S_1. For this test the "*b*" lines were found most favorable. If the slight motion of C_2 to secure this last step displaces the central emergent beam at *F* the prism must be moved enough to restore it and the last step repeated. When the adjustments are finally secured, collimator C_2 is clamped, and prism *AEI* is secured in its position. The instrument is then ready for calibration. The author did this by using the Fraunhofer lines of the solar spectrum, and determining the reading of the scale attached to the telescope when sighted on each of nineteen lines.

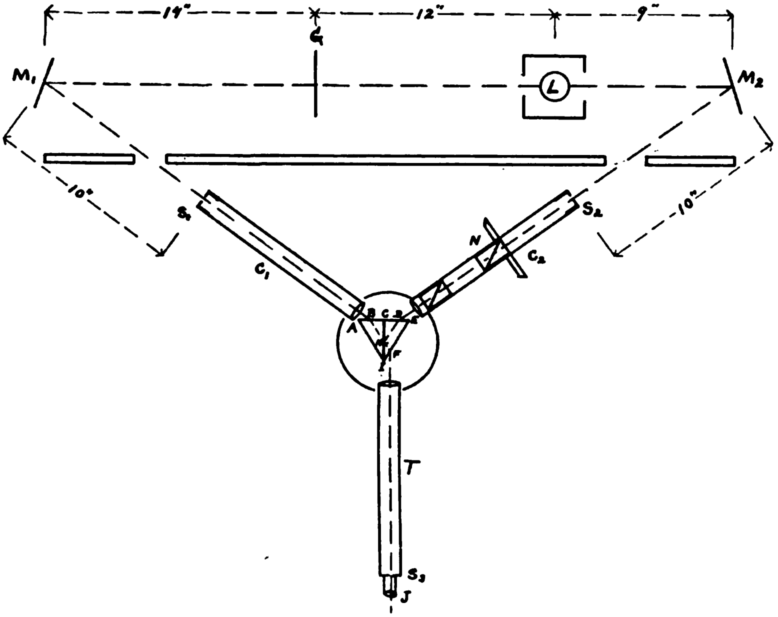

Fig. 1.—Lemon-Brace spectro-photometer

After calibration, the instrument was set up as shown in Figure 1. *L* is a 50-watt, nitrogen-filled, electric lamp of white glass. It is inclosed in a box with openings as shown. M_1 and M_2 are mirrors adjustable about vertical and horizontal axes so as to reflect the light from *L* along the optical axes of C_1 and C_2.

In order that the illumination of the silver strip by M_2 might be brighter than that of the background by M_1 when the nicols were parallel, it was necessary to have L much nearer M_2 than M_1 and also to place a piece of ground glass, G, in position as shown. An improvement could be made by the use of a lamp of higher power with an adjustable rheostat. This would give better illumination at the ends of the spectrum and also enable one to reduce the brightness at the middle of the spectrum. With the present arrangement the ends are too faint for accurate measurement, while the middle is too bright for comfort.

While making measurements, the room was darkened and the lamp, mirror, etc., were covered with black cloth. Slits S_1 and S_2 were opened to a width of ½ mm. Eyepiece J was removed, and slit S_3 made as narrow as possible without producing diffraction fringes. The telescope was set at such a scale-reading as to bring light of the desired wave-length to S_3. Without having the ray-filter in place, five settings of nicol N were made, balancing the illumination of the silver strip against its background. The average $\sin^2$ of these readings gave the fractional part of the light of the chosen wave-length entering S_2 necessary to balance that entering S_1. The ray-filter was then put in front of S_1 and five other settings of N were made. The average sine^2 of these readings gave the fractional part of the light entering S_2 now necessary to balance that transmitted by the ray-filter. The ratio between the two averages gave the fractional part of the incident light of the chosen wave-length which was transmitted by the ray-filter. By making the measurements in this way, the percentage of transmission at any particular wave-length could be obtained at any time with no necessity of taking any other measurements in the series.

The current through the lamp L sometimes dropped to 50 per cent, or less, of its normal value, when the motor raising the floor in the big dome was started, but no perceptible effect on the balancing of illuminations in the Brace prism was produced thereby.

The five ray-filters already described were examined for their color-transmission and the percentages of transmission of the selected wave-lengths are given in the following table. The curves for filters "Dec. '07" and "M 1" appear in Figure 2. It will be seen how closely the filters "Dec. '07" and "M 1" correspond in their transmission to the flat part of the color-curve of the 40-inch objective shown in Figure 3, which is taken from the investigation published by Philip Fox in this *Journal*, **27**, 237, May 1908. The fact that the filters transmit the longer wave-lengths for which the color-curve slopes upward is neutralized by the lack of sensitiveness of the photographic plates to those wave-lengths. The combination of the three elements thus enables the great light-gathering power of the 40-inch telescope to be employed photographically with practically no troublesome effects due to chromatic aberration.

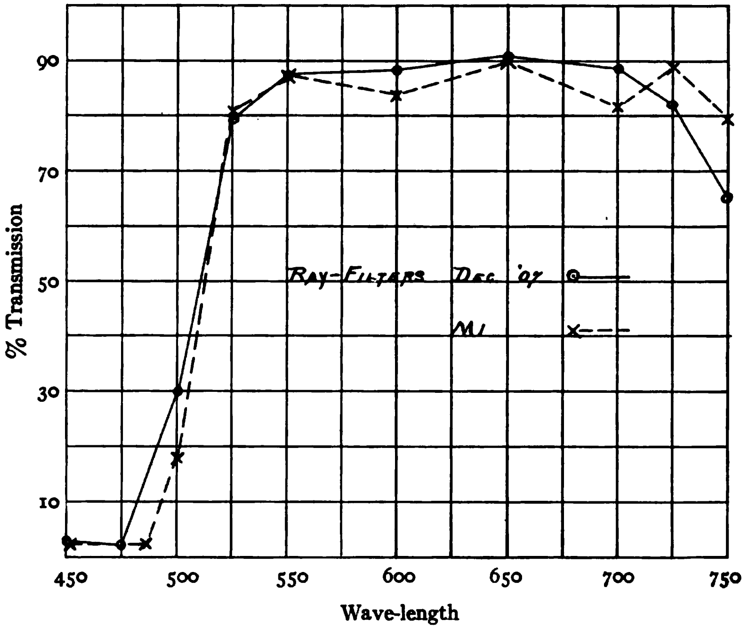

Fig. 2

The ray-filter possessing these transmission characteristics, combined with a plate having a maximum sensitiveness to the yellow, gives a combination whose position of maximum sensitiveness corresponds to that of the retina of the human eye. This is one of the reasons why the determinations of stellar magnitudes

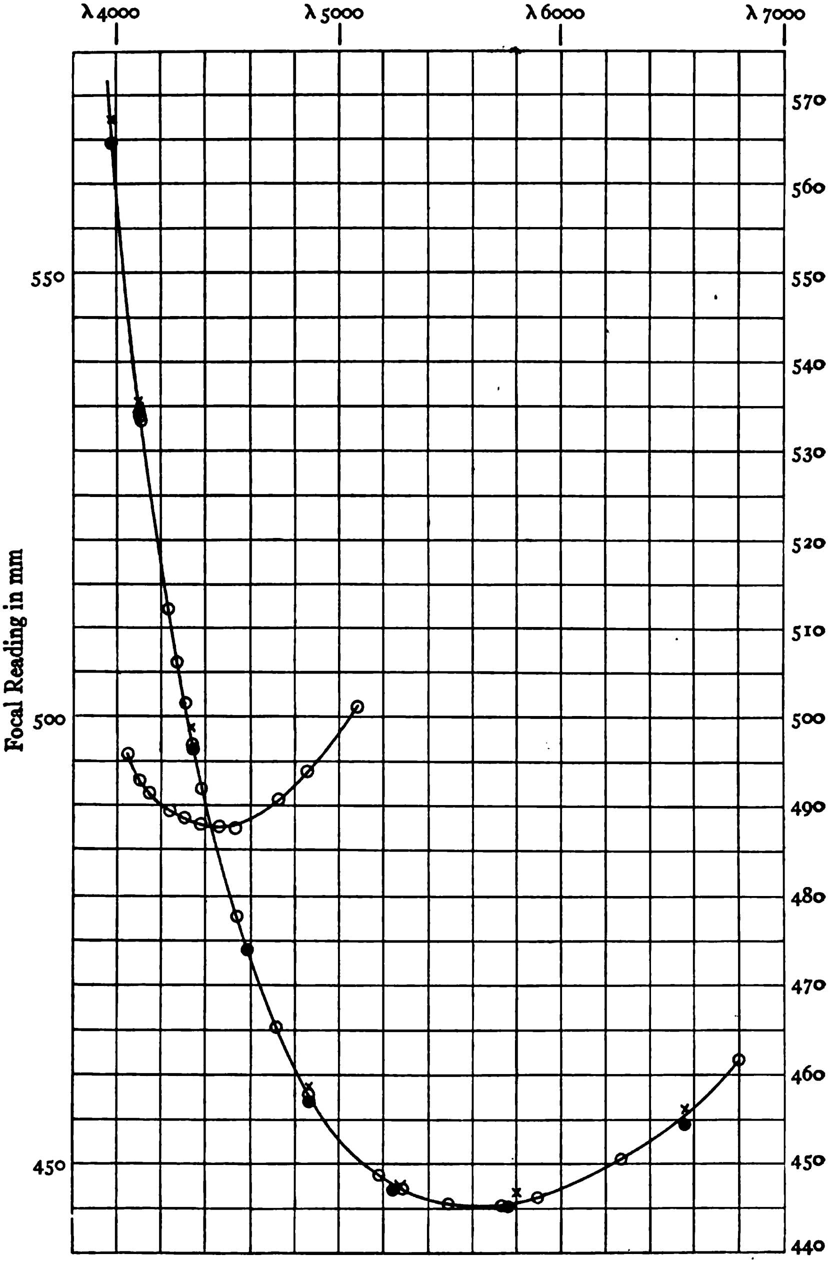

FIG. 3.—Color-curve of the 40-inch objective

made by Professor J. A. Parkhurst by his photographic method have such close correspondence to the visual magnitudes.

TABLE I

TRANSMISSION-PERCENTAGES OF RAY-FILTERS

Wave-Length	J 1	M 1	Beta 10	Dec. '07	Beta 7
750	87.1	79.4	82.0	64.8	72.2
725	85.5	89.2	73.9	81.8	79.3
700	89.3	81.8	74.0	88.7	78.0
650	92.6	90.3	82.2	91.0	87.1
600	87.7	84.0	86.2	88.5	83.5
550	88.7	87.8	88.8	87.6	87.8
525	87.3	81.1	85.3	79.6	81.5
500	57.6	18.5	40.0	29.8	40.8
486	7.9	2.0	14.4	....	....
475	2.5	0.6	6.0	2.0	12.5
450	....	....	....	2.3	4.9

The three colored glasses in use with a Hess-Ives Tint Photometer were also examined for their color-transparency. The results appear in the following table and in Figure 4.

TABLE II

TRANSMISSION-PERCENTAGES OF COLORED GLASSES, HESS-IVES TINT PHOTOMETER

RED GLASS		GREEN GLASS		BLUE-VIOLET GLASS	
Wave-Length	Percentage	Wave-Length	Percentage	Wave-Length	Percentage
766	0.0	628	0.0	495	0.0
750	11.7	610	0.04	486	0.8
725	57.7	590	1.1	475	3.9
700	73.0	570	6.0	465	9.2
675	78.7	550	15.9	455	14.4
650	78.4	530	19.5	445	19.1
625	65.0	510	10.5	435	20.9
600	3.4	490	0.8	425	18.7
489	0.0	468	0.0	415	0.0

It will be seen that the red glass has a maximum transmission of 78.7 per cent for light of wave-length 675 $\mu\mu$. Its average transmission in the region for which it transmits appreciably, i.e., between 766 $\mu\mu$ and 589 $\mu\mu$ is 51.5 per cent of the incident light. In the same way the maximum transmission for the green glass is 19.5 per cent for light of wave-length 530 $\mu\mu$, and its average transmission for the region between 628 and 468 $\mu\mu$ is 6.7 per cent. The blue-violet glass has a maximum transmission of 20.9 per cent for light of wave-length 435 $\mu\mu$, and an average transmission for the region between 495 $\mu\mu$ and 415 $\mu\mu$ of 10.9 per cent. It is very noticeable how little overlapping there is in the light transmitted by either pair of adjacent glasses.

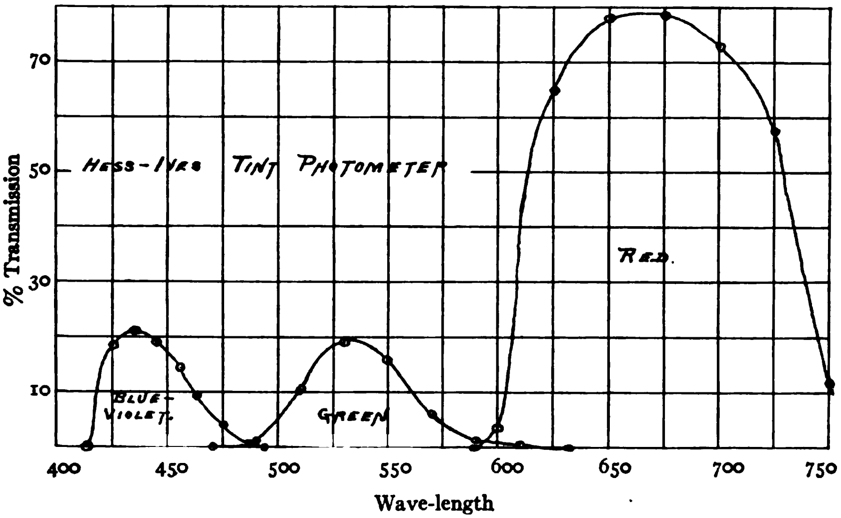

Fig. 4

An interesting piece of incidental information is the relative selective absorption exerted by the nicol prisms and the silver strip. When the readings are taken with no ray-filter in front of S_1, the angle through which the nicol N must be rotated to produce a balancing of illuminations is not the same for all wave-lengths. At first it increases as wave-length decreases, remains practically uniform from $\lambda = 600\ \mu\mu$ to $\lambda = 525\ \mu\mu$, and then decreases with the wave-length, showing that there is selective absorption corresponding in relative values to the $\sin^2$ of the angle of rotation of the nicol. The table below is computed from the average values obtained in the measurements of filters "M 1," "Beta 10," and "J 1."

The measurements on ray-filters "Dec. '07" and "Beta 7" were made after the instrument had been taken down, readjusted and calibrated, and therefore could not be included. Figure 5 gives the curve corresponding to this table.

TABLE III

Wave-length	750	700	650	600	550	525	500	485	475
Sin^2	0.246	.415	.563	.613	.610	.611	.575	.535	.513

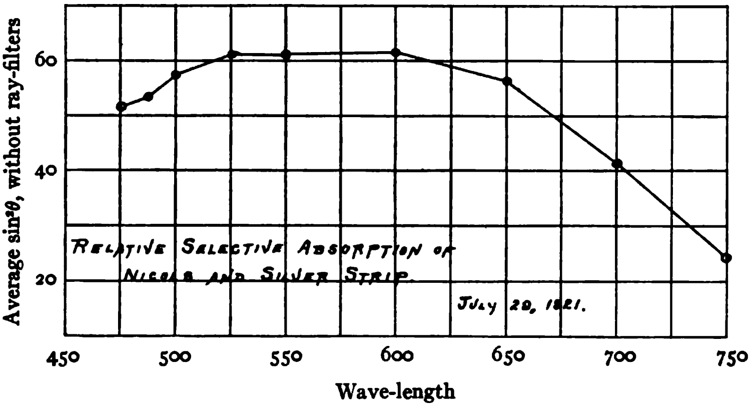

FIG. 5

The author is greatly indebted to Professor J. A. Parkhurst for suggesting the investigation and for his kind assistance during its progress and in the preparation of this report.

MIDDLEBURY COLLEGE
October 1, 1921

SYSTEMATIC CORRECTIONS TO SPECTROSCOPIC AND TRIGONOMETRIC PARALLAXES[1]

By GUSTAF STRÖMBERG

ABSTRACT

Systematic corrections to reduce spectroscopic and trigonometric parallaxes to absolute parallaxes.—The fact that the systematic and accidental errors of spectroscopic parallaxes are proportional to the parallaxes themselves, while the errors in trigonometric parallaxes are independent of the values of the parallaxes, makes it possible, theoretically at least, to determine from a comparison of the two systems the true systematic corrections for each system and even for each observer. The basis of grouping the stars must be independent of each system and therefore Kapteyn's mean parallaxes, derived from apparent magnitudes and proper motions, were used. The factorial correction (x) for the spectroscopic parallaxes and the additive correction (s) for the trigonometric parallaxes, connected by the fundamental equation $x\pi_s=\pi_t+s$, were thus determined. For dwarfs x ranges from 0.97 to 1.07 for different spectral types; for giants the range is from 0.89 to 1.13, the extreme values being rather uncertain. The corresponding systematic correction to absolute magnitudes averages only about 0.1 for dwarfs, but may be somewhat larger for the giants. *For the separate observatories*, McCormick, Allegheny, Yerkes, and Mount Wilson, the reductions from relative to absolute parallaxes come out, respectively, +0″.0053, +0″.0076, +0″.0084 and −0″.0010 within about ±0″.002. These values agree well with those obtained by van Maanen and Miss Wolfe. Hence the list of 1646 spectroscopic parallaxes recently published from Mount Wilson probably gives the absolute parallaxes without any appreciable systematic error.

The large number of spectroscopic parallaxes now available enables us to determine fairly accurate values of the systematic corrections for both spectroscopic and trigonometric parallaxes. In the derivation of the reduction tables for converting line intensities into absolute magnitudes used in the recent list of 1646 parallaxes[2] no corrections were applied to the trigonometric parallaxes, but it was found that if the corrections deduced by van Maanen and Miss Wolfe[3] had been used, the effect due to these corrections would have been less than 0.1 magnitude. The method used by van Maanen and Miss Wolfe consisted in comparing the means of the trigonometric parallaxes found by different observers for groups of stars of nearly the same mean apparent magnitude and mean proper motion. Their corrections represent the deviations of the results by individual observers from the mean system defined by all the trigonometric parallaxes. Since the systematic errors of individual observers must compensate each other to a large extent in the mean system, the corrections of van Maanen and Miss Wolfe are presumably close approximations to the true systematic errors.

---

[1] *Contributions from the Mount Wilson Observatory*, No. 220.

[2] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 1, 1921.

[3] *Mt. Wilson Contr.*, No. 189.

Other determinations of systematic corrections of trigonometric parallaxes have been made by Flint[1] and B. Boss,[2] the former reducing his corrections to a system based upon the Yale parallaxes, the latter comparing the spectroscopic parallaxes published in *Mt. Wilson Contribution*, No. 142 (*Astrophysical Journal*, **46**, 313, 1917) with the different series of trigonometric parallaxes.

The comparison of spectroscopic and trigonometric parallaxes enables us, theoretically at least, to determine the true systematic corrections for each observer and for the spectroscopic system as well. This depends upon the fact that the systematic and accidental errors of the spectroscopic parallaxes are proportional to the parallaxes themselves, while the errors in the trigonometric parallaxes are independent of the size of the parallax. Hence, for any series of trigonometric parallaxes, when compared with the spectroscopic system, we have, as shown more in detail below, an equation of the form

$$x\pi_s = \pi_t + s,$$

from which the systematic correction s, and the correction factor x for the spectroscopic parallaxes, can be determined. The successful use of the equation of course presupposes a considerable range in the values of the parallaxes compared. Another important advantage of the method is that any error in the assumed values of the mean parallax of the comparison stars used for the reduction of the relative trigonometric parallaxes to absolute values goes over into s and is determined as a part of the systematic correction. In principle, therefore, the comparison permits us to establish an absolute system of parallaxes.

The principal object of this investigation being to determine systematic corrections of the spectroscopic parallaxes as a whole, taking into account the possibility of constant corrections to the trigonometric parallaxes, only the larger lists of trigonometric parallaxes, viz., those derived at the McCormick, Allegheny, Yerkes, Mount Wilson, Sproul, and Yale observatories, have been discussed.

---

[1] *Astronomical Journal*, **29**, 189 (No. 696), 1916.

[2] *Ibid.*, **33**, 17 (No. 771), 1920.

The following notation has been used:

M = Absolute magnitude as published in *Mt. Wilson Contribution*, No. 199.
ΔM = Systematic correction to M to obtain the most probable absolute magnitude.
T = Accidental error of $M+\Delta M$.
π = True parallax.
π_1 = Spectroscopic parallax based on the absolute magnitude M.
π_0 = Trigonometric parallax.
s = Systematic correction to π_0
ϵ = Accidental error of π_0+s.

We have then (cf. *Mt. Wilson Contribution*, No. 199, p. 4)

$$\left.\begin{aligned} & & \pi &= 10^{0.2(\Delta M+T)}\pi_1 = \pi_0+s+\epsilon \\ &\text{or} & \sigma\tau\pi_1 &= \pi_0+s+\epsilon \\ &\text{where} & \sigma &= 10^{0.2\Delta M}, \qquad \tau = 10^{0.2T} \end{aligned}\right\} \qquad (1)$$

Expanding τ in a power series,

$$\left.\begin{aligned} \tau &= 1+E+\tfrac{1}{2}E^2 = A+E \\ E &= \frac{0.2T}{\text{Mod}} \\ A &= 1+\tfrac{1}{2}E^2 \end{aligned}\right\} \qquad (2)$$

Thus

$$\sigma A\pi_1 - s - \pi_0 = \epsilon - E\sigma\pi_1 \qquad (3)$$

As the right-hand side of this equation must be assumed equal to zero, we cannot group the stars according to the size of π_0 or π_1. If we group the stars according to values of π_0 the quantity $\bar{E}$ could be assumed zero,[1] but $\bar{\epsilon}$ would show a continuous decrease from positive to negative values as $\bar{\pi}_0$ increases. The same holds for $\bar{E}$ if we group the stars according to π_1, and furthermore the weight of the unknown quantity σ would be systematically affected by the accidental errors in π_1. These circumstances correspond to the difference in slope of the two regression lines in the correlation theory. A neutral basis of grouping according to the size of the parallaxes must therefore be used, i.e., one for which we can assume the errors in the parallaxes to be independent of the errors in the measured quantities M and π_0. Such an independent basis of grouping is afforded by Kapteyn's mean parallaxes, which are a function of apparent magnitude and proper motion. For each star the mean parallax was therefore computed from Kapteyn's formula in *Groningen Publication*, No. 8, and groups of stars were then formed having mean parallaxes within certain limits. The mean spectroscopic and trigonometric parallax was then found for each group. Each of these groups then furnished an equation of condition.

---

[1] A bar above a symbol denotes throughout this paper an algebraic mean.

From equation (3) we find

$$\sigma\bar{A}\bar{\pi}_1 - s - \bar{\pi}_0 = \bar{\epsilon} - \sigma\overline{E\pi_1}\,.$$

For a neutral basis of grouping we have $\bar{\epsilon}=0$ and $\bar{E}=0$. The second term on the right-hand side, however, is not quite zero, as E is to some extent dependent on π_1 and thus $\overline{E\pi_1}$ is not exactly equal to $\bar{E}\bar{\pi}_1$.

To reduce the right-hand side of the equation of condition to zero, we multiply equation (1) by

$$\frac{1}{\tau} = 1 - E + \tfrac{1}{2}E^2 = A - E$$

and find

$$\sigma\pi_1 = A(\pi_0 + s + \epsilon) - E(\pi_0 + s + \epsilon).$$

Thus

$$\frac{\sigma}{A}\pi_1 - s - \pi_0 = \epsilon\left(1 - \frac{E}{A}\right) - \frac{E}{A}(\pi_0 + s).$$

For a neutral basis of grouping, we have, including terms of second order of E,

$$\frac{\sigma}{\bar{A}}\pi_1 - s - \bar{\pi}_0 = \bar{\epsilon}(1 - \bar{E}) - \bar{E}(\bar{\pi}_0 + s) = 0.$$

The equations of condition thus become

$$\left.\begin{aligned} x\pi_1 - s - \pi_0 = 0 \\ \text{Weight} = \frac{1}{\bar{\epsilon}^2 + \bar{E}^2\bar{\pi}^2} = \frac{N}{\overline{\epsilon^2} + \overline{E^2}\,\bar{\pi}^2} \end{aligned}\right\} \qquad (4)$$

TABLE I

SYSTEMATIC CORRECTIONS—SEPARATE SOLUTION FOR EACH TYPE

Spectrum	M	x	McCormick s_1	Allegheny s_2	Yerkes s_3	Mount Wilson s_4	Sproul s_5	Yale s_6
A6 to F9....	1.0 to 5.5	1.056	+0″.0037 (53)	+0″.0020 (92)	+0″.0013 (25)	−0″.0009 (24)	−0″.0068 (35)	−0″.0008 (46)
		± .035	± .0020	± .0015	± .0019	± .0015	± .0021	± .0022
G0 to G9....	≦3.0	1.095	+0.0012 (35)	+0.0011 (69)	+0.0052 (10)	−0.0040 (13)	−0.0114 (11)	+0.0085 (11)
		± .134	± .0033	± .0031	± .0055	± .0036	± .0062	± .0082
G0 to G9....	≧3.1	0.975	−0.0055 (36)	+0.0038 (60)	−0.0048 (27)	+0.0221 (8)	−0.0130 (10)	−0.0008 (74)
		± .053	± .0044	± .0032	± .0034	± .0040	± .0054	± .0031
K0 to K9....	≦4.0	1.075	−0.0028 (25)	+0.0058 (54)	+0.0181 (7)	−0.0085 (26)	−0.0090 (8)	+0.0198 (7)
		± .113	± .0034	± .0025	± .0050	± .0017	± .0040	± .0056
K0 to K9....	≧4.1	1.030	+0.0040 (39)	+0.0069 (27)	+0.0109 (30)	−0.0012 (7)	+0.0133 (20)	+0.0054 (70)
		± .064	± .0061	± .0062	± .0052	± .0092	± .0067	± .0050
A6 to K9....	....	....	+ .0012 (188)	+0.0030 (302)	+0.0026 (99)	−0.0024 (78)	−0.0069 (84)	+0.0018 (208)
			± ″.0016	± ″.0011	± ″.0014	± ″.0010	± ″.0016	± ″.0016

where $x = \frac{\sigma}{A}$ is the quantity by which the spectroscopic parallaxes must be multiplied in order to obtain the best agreement with π_0+s. N is the number of stars in the group.

Assuming only that the quantities x and s are constant for a certain group of stars, we can determine them separately without introducing any limitation as to their size. The value of s found in this way is the systematic correction which reduces π_0 to *absolute* parallax.

We have assumed that the external accidental probable error in the trigonometric parallaxes determined at the McCormick, Allegheny, Yerkes, and Mount Wilson observatories is $\pm 0''.010$. The probable errors of the spectroscopic parallaxes have been assumed to be $\pm 0.20\,\pi_1$. The weights of the equations of condition are therefore proportional to $N/(0.09\pi^2+0.000225)$ where for $\bar{\pi}$ we have used $\frac{1}{2}(\bar{\pi}_1+\bar{\pi}_0)$, x being nearly equal to unity.

The following provisional reductions to absolute parallaxes were applied before the computations were made: For McCormick, Allegheny, and Yerkes observatories, $+0''.005$; for Mount Wilson, $+0''.002$; for Sproul and Yale the reductions were computed from the table given by Kapteyn in *Groningen Publications*, No. 24, page 15. The Sproul and Yale parallaxes were given half-weight.

The value of x was determined separately for giant and dwarf stars of different spectral types. For the first solution the systematic correction s was determined for each of the above-mentioned groups of trigonometric parallaxes. The results are given in Table I. The first column indicates the limits of spectral type, the second the division according to absolute magnitude. The probable errors for x and s are below the numbers to which they belong. The numbers of stars used are given in parentheses. The last line shows the weighted means of s for all types and absolute magnitudes.

Using these mean systematic corrections to the trigonometric parallaxes and recomputing the correction factor of the spectroscopic parallaxes, we find the values of x and ΔM_1, given in Table II. The quantity ΔM_1 is the correction that must be applied to M in

TABLE II

CORRECTION FACTORS FOR SPECTROSCOPIC PARALLAXES

Spectrum	M	x	ΔM_1
A6 to F9	1.0 to 5.5	1.059±.012	+0.12±0.02
G0 to G9	≦3.0	1.121±.040	+0.25±0.08
G0 to G9	≧3.1	0.999±.029	0.00±0.06
K0 to K9	≦4.0	1.001±.059	0.00±0.13
K0 to K9	≧4.1	0.960±.019	−0.09±0.04
F0 to F9	≦0.0	0.149±.103	+0.30±0.20
Ma to Md	<3.0	0.892±.081	−0.25±0.20
Ma to Md	>7.0	1.089±.086	+0.18±0.17

order to obtain the most probable parallax and is determined by the equation

$$\Delta M_1 = 5 \log x = \Delta M - 5 \log \bar{A}.$$

Another solution was made, using all the spectral types, A6 to Md, and solving all the equations as a simultaneous set of conditions, assuming the correction s to be constant for each observer. The results of this solution are given in Table III.

TABLE III

SIMULTANEOUS SOLUTION FOR ALL TYPES

	s		s
McCormick	−0″.0015±0″.0018	Mount Wilson	−0″.0038±0″.0014
Allegheny	+ .0010± .0015	Sproul	− .0085± .0025
Yerkes	+ .0019± .0021	Yale	+ .0004± .0021

Spectrum	M	x	ΔM_1
A6 to F	1.0 to 5.5	1.015±.040	+0.03±0.08
G	≦3.0	1.037±.078	+0.08± .16
G	≧3.1	0.969±.033	−0.07± .07
K	≦4.0	0.922±.074	−0.18± .17
K	≧4.1	0.940±.029	−0.14± .07
F	≦0.0	0.962±.247	−0.08± .57
M	<3.0	0.750±.154	−0.63± .45
M	>7.0	1.082±.045	+0.17± .09

From van Maanen's investigation we know, however, that the Sproul and Yale parallaxes have a considerable magnitude correction. Such an error would affect appreciably the determination of the factor x, and for this reason a final solution was made, omitting the Sproul and Yale parallaxes. The results of this solution, which are given in Table IV, are probably the most trustworthy. They are in good agreement with those found by van Maanen and Miss Wolfe, which, reduced to the same system of corrections for reduction to absolute parallaxes as used here, are given in the last column of Table IV.

TABLE IV

SIMULTANEOUS SOLUTION (SPROUL AND YALE OMITTED)

	s	s (van Maanen)
McCormick..............	+0.″0003±0.″0021	−0.″0016
Allegheny................	+ .0026± .0017	+ .0027
Yerkes..................	+ .0034± .0022	+ .0036
Mount Wilson............	− .0030± .0014	− .0020

Spectrum	M	x	ΔM_1
A6 to F.......	1.0 to 5.5	1.041±.045	+0.09±0.09
G............	≦3.0	1.135±.092	+0.27± .18
G............	≧3.1	0.987±.037	−0.03± .08
K............	≦4.0	1.012±.086	+0.03± .18
K............	≧4.1	0.972±.034	−0.06± .08
F............	≦0.0	1.090±.254	+0.19± .52
M............	<3.0	0.888±.167	−0.26± .41
M............	>7.0	1.068±.052	+0.14± .11

Using the systematic corrections resulting from this final solution, we find the following reductions from *relative* to *absolute* parallax.

McCormick	+0.″0053±0.″0021	Yerkes	+0.″0084±0.″0022
Allegheny	+0.″0076±0.″0017	Mount Wilson	−0.″0010±0.″0014

The systematic corrections to the absolute magnitudes are in general very small except in the case of the giant G and M stars and the very brightest F stars, most of which are Cepheids or pseudo-Cepheids. For these stars, however, the probable errors are rather large on account of the smallness of the parallaxes. In the previous derivation of the reduction tables for the determination of absolute magnitudes it was found that in the case of the giant G stars the results from trigonometric parallaxes and parallactic motion differed considerably. The determination of the mean absolute magnitudes based on parallactic motion would make these stars considerably brighter than the results from the trigonometric parallaxes indicate. A weighted mean was accordingly used. The discordance is now nearly eliminated by the application of the systematic corrections to the trigonometric parallaxes. The same holds for the brightest F stars, for which originally the measured parallaxes were not used at all, the result from parallactic motion having much higher weight. For the giant M stars the original determination was based mainly on the peculiar motion, and the present results seem to indicate that for these stars as a whole the system is nearly correct.

As the final result of this investigation, we may say that the system of spectroscopic parallaxes in *Mt. Wilson Contribution*, No. 199, as a whole, is correct within the errors of the quantities involved, and that there is no need at present to apply corrections to the system. The systematic corrections to the trigonometric parallaxes determined at the McCormick, Allegheny, Yerkes, and Mount Wilson observatories have been found to agree satisfactorily with those of van Maanen and Miss Wolfe, and, as the method used here gives absolute corrections, it can be regarded as probable that after these corrections have been applied there remain no appreciable outstanding constant errors in the resulting absolute parallaxes.

MOUNT WILSON OBSERVATORY
September 1921

NEW MEASUREMENTS OF STELLAR RADIATION

BY W. W. COBLENTZ

ABSTRACT

Relative total radiation of 27 bright stars, down to magnitude 3.8.—New measurements which were recently made with the 40-inch reflector of the Lowell Observatory at Flagstaff, using a vacuum thermo-couple, are given in a table. These confirm the result obtained at Mount Hamilton in 1914 that the total radiation from red stars is 2.5–3 times as much as that from blue stars of the same visual magnitude.

Distribution of energy in the spectra of some bright stars was determined by the use of a series of filters which isolated various spectral regions. In this *preliminary note* merely the general result is reported that the *maximum emission* lies in the infra-red (0.7–0.9 μ) for stars of types K and M, and in the ultra-violet for stars of types B and A. The corresponding *black body temperatures* vary from 3000° C., for the red M stars to 9000° or 10,000° C., for the blue B stars. The *percentage transmission through 1 cm of water* is given for 22 stars and varies from 81 for B stars to 40 for M stars.

In a previous paper[1] data were given on a comparison of stellar radiometers and radiometric measurements of stars as observed at an altitude of about 4000 feet at Mount Hamilton, California, with the Crossley 36–inch reflector of the Lick Observatory. Quantitative measurements were made of stars down to magnitude 5.3 and qualitative measurements to magnitude 6.7. It was found that red stars emit from 2.5 to 3 times as much *total* radiation as blue stars of the same visual magnitude.

These observations were verified by an independent method which consisted in measuring the transmission of stellar radiation through a 1-cm cell of water, having quartz windows. By this means it was shown that, of the *total* radiation emitted, blue stars have about two times as much *visible* radiation as yellow stars, and about three times as much *visible* radiation as red stars.

At various times during the past seven years attention was given to the improvement of the stellar radiometers,[2] galvanometers, etc. The various subsidiary data will be published in a complete paper dealing with the whole subject.

The object of this paper is to give a preliminary survey of the results of new stellar radiometric measurements, made at a much higher altitude, 7300 feet, at Flagstaff, Arizona, with the 40–inch reflector of the Lowell Observatory. This was made possible through the generosity of Drs. V. M. Slipher and C. O. Lampland. Not only was an otherwise busy program interrupted, but furthermore, Dr. Lampland personally operated the telescope, thus insuring speed and efficiency in accomplishing results. It is a pleasure to record here my grateful acknowledgments for the many courtesies accorded by various members of the staff of the Lowell Observatory.

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[1] Coblentz, *Bulletin of the Bureau of Standards*, 11, 613, 1914.

[2] *Bulletin of the Bureau of Standards*, 13, 423, 1916; 14, 532, 1918; 16, 253, 1920; *Journal of the Washington Academy of Sciences*, 6, 473, 1916.

The object of the present investigation was (1) to verify previous results; (2) to measure the intensities of radiation of bright stars in the region of 0 hours to 12 hours in right ascension, not previously measured; and (3) to determine the feasibility of the method of obtaining the distribution of spectral energy of stars by means of transmission screens which, either singly or in combination, are placed in front of the vacuum thermo-couple. By means of these thermo-couples, measurements were made on the intensities of radiation of thirteen bright stars not observed in 1914, thus completing the survey of the whole sky. A total of thirty celestial objects was measured, including Venus and Mars.

By means of a series of transmission screens (of yellow and red glass, of water, and of a thick plate of quartz) wide spectral regions were isolated and the intensities of radiation in the spectrum from 0.3 μ to 0.43 μ; 0.43 μ to 0.6 μ; 0.6 μ to 1.4 μ; 1.4 μ to 4 μ; and 4 μ to 10 μ were determined. In this manner the distribution of energy in the spectra of sixteen stars was determined, and thus was obtained for the first time an insight into the intensities of radiation in the complete spectrum of a star.

By means of this device it was found that in the stars of types B and A the maximum intensity of radiation lies in the ultra-violet (0.3 μ to 0.4 μ) while in the cooler stars of types K and M the maximum emission lies at 0.7 μ to 0. 9 μ in the infra-red. From this it appears that the black-body temperature (i.e., the temperature which a black body would have to attain in order to emit a similar distribution of relative spectral energy) varies from 3000°C. for red M stars to 9000° or 10,000°C. for blue B stars.

The observing station being much higher than previously used, the atmospheric scattering of light was greatly reduced and consequently the transmissions in the violet are somewhat higher than previously observed when the water-cell was interposed.

TABLE I

TRANSMISSION OF STELLAR RADIATION THROUGH A 1-CM LAYER OF WATER

Star	Magnitude Harvard Revision	Deflection for a Galvanometer Sensitivity of $i = 1 \times 10^{-10}$ Amperes	Spectral Type	Percentage of Transmission	Remarks
θ Orionis (Nebula)	....	±0.02 cm	......	......	
ε Orionis........	1.75	0.66	B	81	
γ Orionis........	1.70	0.64	B2	82	
α Leonis.........	1.30	0.77	B8	......	
β Tauri..........	1.78	0.78	B8	79	
β Canis Minoris..	3.09	0.38	B8	......	
β Orionis........	0.34	2.89	B8p	63*	Rigel, spectroscopic binary
β Lyrae.........	Var.	0.26	B8p	......	
α Geminorum....	1.99 2.85	0.78	A	82	Castor
α Lyrae.........	0.14	3.60	A	75	Vega
γ Geminorum....	1.93	0.71	A	......	
α Canis Majoris..	−1.58 8.5	10.62	A	65*	Sirius, binary
α Cygni.........	1.4	1.34	A2	76	Deneb
α Aquilae........	0.9	1.88	A5	71	Altair
α Persei.........	1.90	0.75	F5	74	
α Canis Minoris..	0.48 13.5	2.67	F5	64*	Procyon, binary
α Aurigae........	0.21	4.91	G	57	Capella
ε Geminorum....	3.18	0.58	G5	66	
β Geminorum....	1.21	2.19	K	58	Pollux
α Boötis.........	0.24	8.10	K	47	Arcturus
γ Leonis.........	2.61 3.80	0.99	K	......	
α Tauri..........	1.06	6.03	K5	42	Aldebaran
λ Aquarii........	3.84	0.75	Ma	47	
μ Geminorum....	3.19	2.11	Ma	41	
β Andromedae....	2.37	2.45	Ma	41	
α Orionis........	0.92	15.00	Ma	34	Betelgeuse
α Scorpii........	1.22	8.82	Map	33*	Antares, spectroscopic binary
β Pegasi.........	2.61	2.96 cm	Mb	38	

However, all the data verify previous measurements showing that blue stars emit less infra-red radiation than do red stars of the same visual magnitude. Moreover, observations made on the same night (same weather conditions) are consistent in showing small gradations in the infra-red component of the radiation, corresponding with the small gradations (say B2 and B8) in spectral types.

For binary stars having companions of low luminosity, transmissions of the water-cell are low, indicating that the companion stars emit considerable infra-red radiation. See Table I.

It was found that even in red stars the component of spectral radiation of wave-lengths greater than 4 μ is only from 1 to perhaps 10 per cent of the total. From this it would appear that in future work it may be permissible to use vacuum thermo-couples with thin quartz windows instead of fluorite, thus saving expense and possible leakage by constructing the container of quartz.

Washington, D.C.
October 24, 1921

SOLAR CONSTANT, SUN-SPOTS, AND SOLAR ACTIVITY

BY ANDERS ÅNGSTRÖM

ABSTRACT

Variation of solar constant with number of sun-spots.—A study of 205 observations for 1915–1917 furnished by C. G. Abbot suggests that the solar constant S does not increase regularly with any power of the sun-spot number N, but rather reaches a maximum value for N between 100 and 160, approximated in accordance with the formula $S=1.903+0.011\sqrt{N}-0.0006N$. The number of observations for high value of N is, however, too small to be decisive.

A comparison of the values of the solar constant computed by Abbot and his collaborators with the sun-spot numbers given by Wolfer shows, if mean values for the separate years are considered, a close connection between the two phenomena. Thus a high sun-spot number seems to correspond to a high value of the solar constant and vice versa. The relation is apparently not a linear one; and inquiring into the exponent which applied to the sun-spot numbers gives the highest correlation between the two phenomena, I have found this exponent to be very nearly equal to $\frac{1}{2}$.[1]

If we consider only yearly mean values, the solar constant S seems then to be given with good approximation by the relation:

$$S=1.903+0.0055\sqrt{N}\,, \qquad (1)$$

where N is the number of sun-spots according to Wolf-Wolfer. The mean difference between observed and computed values during the epoch 1905–1917 is less than 0.01, i.e., less than 0.5 per cent of the value of the solar constant itself, as may be seen from Table I.

As the question regarding the probable connection between solar constants and sun-spot numbers is an important one for the solution of many problems connected with the climate of past epochs and with the constitution of the sun, it seems worth while to enter a little more closely into the question.

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[1] *Geografiska Annaler*, H. 1, 1920.

For the low values of N which are obtained by taking averages during a considerable time the relation (1) seems to hold with good approximation. But is this true also for a wider range of values of N? If we consider the values of S and N corresponding to individual days, what will be the relation between them?

In order to answer these questions, I have divided the values of the solar constant during the years 1915–1917, kindly put at my disposal by Dr. Abbot, in groups according to the corresponding sun-spot numbers, as is seen from Table II. The years named are selected from the point of view that the sun-spot numbers have varied during that time over a considerable range ($0<N<256$), and also because the variation in the yearly mean values is small, and consequently should not have introduced any complications. The values of the solar constant corresponding to the various groups of sun-spot frequencies are given in Table II together with the numbers of values of which they represent the means. On the basis of this table, Figure 1 has been drawn, where the values of the solar constant are plotted against sun-spot numbers.

The table and figure seem now to suggest the following conclusions. With increasing sun-spot numbers, the solar constant first increases in order to reach a maximum, which seems to correspond to a sun-spot number of between 100 and 160 (maximum difficult to locate exactly); thereafter, the solar constant decreases with an increase in sun-spot numbers. The relation between solar constant and sun-spot numbers is apparently governed by two separate phenomena. The one causes an increase of the solar constant with increasing sun-spot numbers, and this effect predominates for the small values of N; the other effect goes in the opposite direction, and predominates for large values of N. Let us start from the assumption that the relation between solar constant and sun-spots for small values of N may be given by the expression

$$S=1.903+k\sqrt{N}, \qquad (2)$$

obtained in its general features by the study of the yearly mean values. We may assume the other effect, which evidently must depend upon a function of higher order of N than $N^{\frac{1}{2}}$, to be of the form N^a, where a is to be determined from the observations. The expression for the relation then takes the form:

$$S = 1.903 + k\sqrt{N} - cN^a. \tag{3}$$

A comparison with the tabulated values gives us the following approximate values for the constants:

$k = 0.011$, $c = 0.0006$, $a = 1$ (approx.).

The smooth curve of Figure 1 is computed from (3) on the basis of these values of a, k, and c.

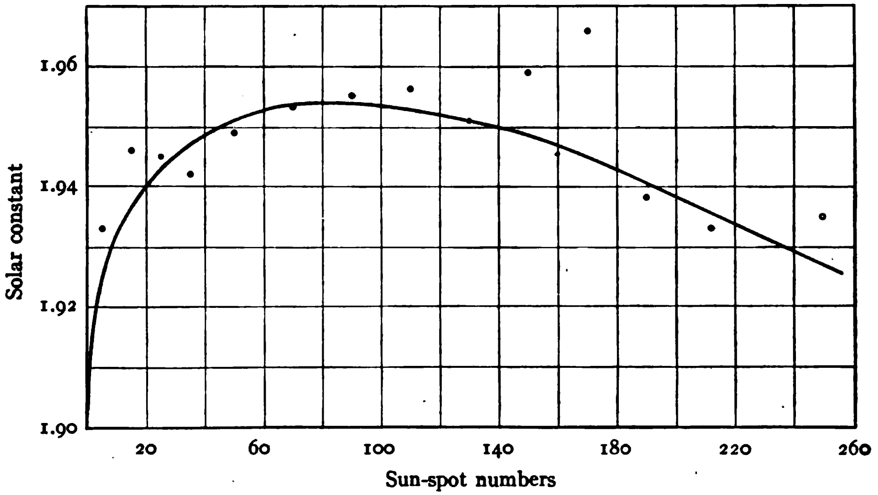

Fig. 1.—Solar constant and sun-spot numbers

It is clear that too much weight ought not to be attached to a result based upon a number of observations which, especially for large values of N, is very limited. In fact, the decrease of the solar constant with increasing N after a maximum value is reached is based upon ten observations, of which, according to Abbot, four are excellent (*e*), two very good (*vg*), three good (*g*), and one poor (*p*). It is to be noted, however, that the value marked (*p*) is the highest one of them all, namely 1.971, and consequently it has raised and not lowered the mean value. But, on the other hand, the result seems to me to be in remarkable agreement with consequences to be expected from the nature and constitution of sun-spots. The sun-spots themselves are dark relative to the surrounding parts of the sun's disk. The decrease in brightness varies between wide limits, but according to E. Liais[1] and M. Gouy and L. Thollon,[2] the larger sun-spots generally have a brightness less than 20 per cent of the normal. The percentage of heat radiation is somewhat larger. S. P. Langley[3] gives it as about one-half of the normal for the photosphere. It is evident that the spot-covered area itself cannot contribute to the increase in the solar constant. The observations of individual sun-spots show, however, that the dark area of the spot generally is surrounded by an outer region, where the brightness for a number of special lines is more intense than the normal for the surface. At this area the protuberances and flocculi are especially strongly developed, surrounding the inner and darker part by a kind of luminous corona of more or less symmetric form. If we assume the point of view expressed by Abbot, Fowle, and Aldrich, namely, that the increase in the solar constant is caused mainly by an increase in the vertical convection through which hot material is carried to the surface of the sun, we must conclude that the seat of this "intensified" radiation must be located especially at peripheric rings round the sun-spots. It seems very natural, then, that the areas of these rings must increase less rapidly than the area of the sun-spots, viz., the Wolf-Wolfer sun-spot numbers. On the other hand, it seems as natural that the simultaneous decrease of the solar constant is proportional to the spot area, which as we know represents dark regions of the sun's surface for luminous as well as dark heat radiation. The measured area of the sun-spots is, it is true, scarcely more than a few thousandths of the sun's disk. But to this area we must probably add a great part of smaller spots of granular structure which never are counted as real sun-spots, but the area of which increases and decreases with the sun-spot number itself.

The data available at present are too few to give a very definite idea of the form of the relation between solar constant and sun-spot number. It is with all probability a function dependent not only on the area of sun-spots, but also on their number, intensity, and constitution in general. *So much seems clear, however, from the previous discussion, that this function is not a linear one; the opinion that an increase of sun-spots is also accompanied by an increase in the solar constant must be subjected to a revision. In all probability the solar constant passes through a maximum with increasing sun-spot area*, after which a continuous decrease of the solar constant takes place when the area increases.

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[1] *Mémoires de la Société des Sciences de Cherbourg*, **12**, 1866.

[2] *Comptes Rendus*, **95**, 1834–1836, 1882.

[3] *Monthly Notices*, **37**, 5, 1876.

From our material at present it therefore does not seem permissible to assume that very high values of the solar constant in the past have accompanied a very high value of the frequency of sun-spots, but, on the contrary, that the solar constant at certain epochs possibly may have had a very low value on account of a great frequency of sun-spots. If we dared to extrapolate the function given above to represent the conditions at very high sun-spot frequencies, we should find that a sun-spot number of 1,000 corresponds to as low a solar constant as about .1.66, or a decrease from the normal by about 15 per cent. As such high sun-spot frequencies have not yet occurred, so far as we have observed, we then enter into the region of hypotheses.

Further determinations of the solar constant, especially at the times of maxima of sun-spots, will be of great interest in throwing more light upon the questions raised above. The foregoing discussion seems to the author to form an additional support for the opinion that the variations of the solar constant, observed by Abbot and his collaborators, are real and the sun consequently is a variable star.

Recently doubts as to the reality of the variations of the solar constant have been raised by G. Granqvist on account of a correlation found to exist between the values of the solar constant and the values of the atmospheric transmission[1]. In a recently published paper[2] I consider that I have shown: (1) that in the case of Abbot's values the correlation pointed out by Mr. Granqvist

[1] *Kosmos*, Stockholm, 1921.

[2] "Är solens strålning variabel?" *Tidskrift för elementär matematik fysik och kemi.* Alb. Bonnier, Stockholm, 1921.

practically disappears, if one accounts for the probable variation of the solar constant itself during the time of observations; and (2) that the connection between solar constant and transmission in the case of the observations of Mr. Granqvist probably is caused by variations in the atmospheric transmission during the time of observations, i.e., too high a value of the solar constant is obtained when, in view of the increasing transmission, too low a value is computed for the transmission, and vice versa.

TABLE I

YEARS	SOLAR CONSTANT (S)		SUN-SPOT NUMBERS (N)	O.–C.
	Observed	Computed	(Wolfer)	
1905	1.956	1.946	63	+0.010
1906	1.942	1.945	58	−0.003
1908	1.936	1.944	55	−0.008
1909	1.918	1.940	46	−0.022
1910	1.921	1.928	21	−0.007
1911	1.921	1.922	3	−0.001
1913	1.904	1.909	1	−0.005
1914	1.956	1.919	9	+0.037
1915	1.952	1.946	62	+0.006
1916	1.946	1.942	50	+0.004
1917	1.960	1.961	113	−0.001
Mean difference	....	....	....	0.009

TABLE II

Sun-Spot Number (N)	Mean Solar Constant (S)	Number of Observations (n)
0–9	1.933	5
10–19	1.946	12
20–29	1.945	17
30–39	1.942	17
40–59	1.949	35
60–79	1.953	33
80–99	1.955	27
100–119	1.956	23
120–139	1.951	11
140–159	1.959	10
160–179	1.966	5
180–199	1.938	2
200–224	1.933	3
225–275	1.935	5

STOCKHOLM
May 1921

THE ORBITS OF THE SPECTROSCOPIC BINARIES 1 HYDRAE AND 75 CANCRI[1]

By R. F. SANFORD

ABSTRACT

Orbits of the spectroscopic binaries, Boss 2227 = 1 Hydrae and Boss 2447 = 75 Cancri.—These stars are of classes F1 and G2 and of visual magnitudes 5.7 and 6.0, respectively. Twenty-seven spectrograms of each give for the periods, 1.563 and 19.46 days; for e, 0.05 and 0.206; for K, 30.3 and 20.2 km/sec.; and for γ, +71.3 and +12.3 km/sec., respectively. Attention is called to the large γ for Boss 2227, and to the fact that Boss 2447 is a dwarf star. Radial velocity-curves are shown.

A previous paper[2] discusses the orbits of seven spectroscopic binaries; the present note gives the orbits of two additional binaries. The general remarks which precede the discussions of individual stars in the first paper are equally applicable here and need not be repeated.

The data in Table I, except for the last column which gives the number of revolutions of the star in its orbit between the first and last observations, are taken from the list of spectroscopic parallaxes of 1646 stars.[3] Figures 1 and 2 show the radial velocity-curves. In the following pages the derivation of the orbits of these two spectroscopic binaries is discussed.

TABLE I

Name	Vis. Mag.	α (1900)	δ (1900)	Spectral Class	Vis. Abs. Mag.	μ	$\pi_{sp.}$	No. Rev.
Boss 2227–1 Hydrae	5.7	$8^h19^m.6$	$-3°26'$	F1	+3.6	$0''.216$	$0''.038$	340
Boss 2447–75 Cancri	6.0	9 2.9	+27 3	G2	+4.1	0.404	0.042	59

BOSS 2227

Measures of the first two spectrograms of this star by Professor H. C. Wilson showed that its radial velocity is variable. Accordingly observations were started immediately for the determination of the elements of its orbit. Twenty-seven suitable spectrograms are listed in Table II. The derivation of the correct period presented some difficulty at first because of its shortness. Finally, all observations were gathered into one period in a satisfactory manner when $P=1.562975$ days was used. Since approximately 340 revolutions of the star in its orbit occurred between the first and last observations it seemed needless to endeavor to correct the period, and therefore the value given above has been taken as final.

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[1] *Contributions from the Mount Wilson Observatory*, No. 221.

[2] *Mt. Wilson Contr.*, No. 201; *Astrophysical Journal*, 53, 201, 1921.

[3] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, 53, 13, 1921.

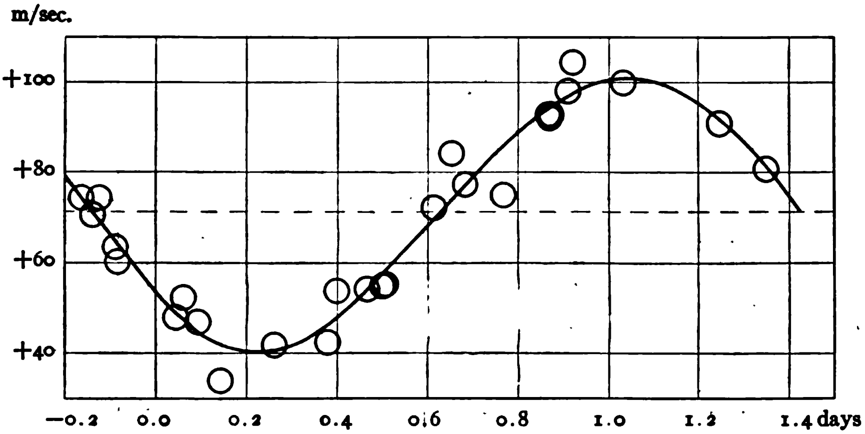

FIG. 1.—Radial velocity-curve for Boss 2227

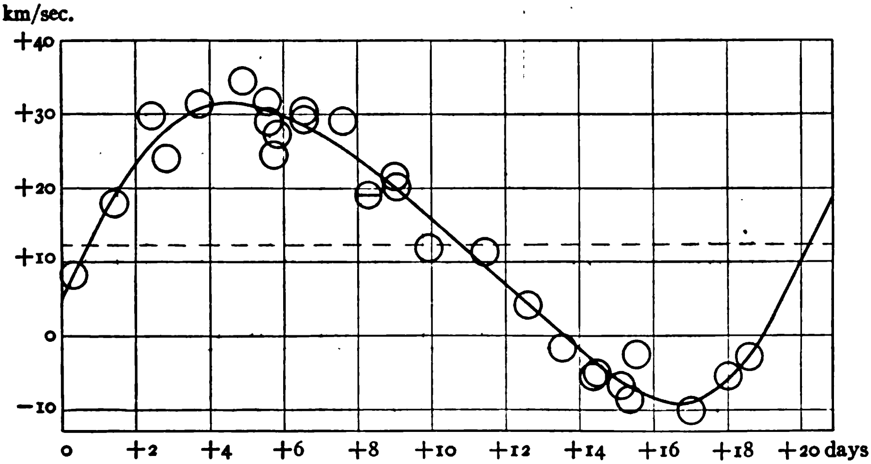

FIG. 2.—Radial velocity-curve for Boss 2447

With preliminary elements derived by Russell's method, an ephemeris was computed and the residuals (O–C) derived. The spectrum is of Class F1, with only one set of lines which at best yields a velocity of only fair precision and which varies so in quality from plate to plate that it has seemed advisable to assign weights to the measures of different plates. The best have been given weight unity and the rest weight 0.75 to 0.50, according to the inherent quality of the lines, or their quality as affected by photographic density, focus, etc., or a combination of these circumstances. The twenty-seven observations were converted into seventeen normal places, assigned proper weights and then used to correct the preliminary elements by the method of least squares. With the elements thus corrected the quantity Σpv^2 is about 75 per cent

TABLE II

OBSERVATIONS OF BOSS 2227

Plate No.	Date	G.M.T.	Phase	Weight	Velocity	O−C
					km/sec.	km/sec.
γ 8860	1919 Nov. 12	$0^h 43^m$	$0^d.062$	0.75	+ 52.2	+ 4.9
9109	1920 Apr. 4	15 11	0.870	0.75	+ 92.6	− 1.8
9666	Oct. 26	0 19	0.501	1.00	+ 54.6	− 2.7
9687	29	0 24	0.380	0.75	+ 42.4	− 3.9
C 741	Nov. 1	0 40	0.262	0.75	+ 41.8	+ 1.3
γ 9705	2	0 13	1.246	1.00	+ 90.6	− 1.2
9723	20	0 36	0.506	0.50	+ 55.0	− 2.8
9732	20	23 58	1.479	0.75	+ 60.3	− 3.6
9738	21	22 47	0.867	1.00	+ 92.8	− 1.4
9747	23	1 06	0.401	0.50	+ 53.7	+ 5.8
C 778	24	1 28	1.416	1.00	+ 70.9	− 0.8
γ 9809	Dec. 22	22 55	0.614	1.00	+ 72.1	+ 2.4
9893	1921 Feb. 15	16 47	0.653	0.50	+ 84.0	+ 9.9
9897	17	15 22	1.031	0.75	+ 99.9	− 0.7
9934	21	18 20	0.466	0.75	+ 54.1	+ 0.4
C 897	22	18 29	1.473	1.00	+ 63.5	− 1.2
γ 9940	23	18 32	0.910	1.00	+ 98.0	+ 1.2
9957	25	19 46	1.400	0.75	+ 74.2	− 0.2
9995	Mar. 16	14 50	1.439	1.00	+ 74.4	+ 5.2
9999	16	18 52	0.045	1.00	+ 48.0	− 0.9
10003	17	14 42	0.871	1.00	+ 92.8	− 1.6
10012	19	15 42	1.349	0.75	+ 80.6	− 0.2
10046	24	15 19	0.095	1.00	+ 46.9	+ 1.8
10091	Apr. 17	16 25	0.682	0.50	+ 77.3	+ 0.1
10099	18	17 03	0.143	0.50	+ 33.9	− 8.0
10124	25	17 40	0.920	1.00	+104.4	+ 7.0
10139	28	17 04	0.768	0.75	+ 75.0	−11.0

of its value when derived with the preliminary elements. The residuals obtained by substitution of the unknowns in the equations of condition agree satisfactorily with the residuals from an ephemeris based on the final elements. In Table III are to be found preliminary elements, corrections, final elements, and their probable errors. The probable error of a single observation of velocity of weight unity is ±2.64 km/sec. The absolute magnitude (+3.6) is very nearly that of Class F stars of greatest frequency and is therefore typical.

TABLE III

Preliminary Elements		Corrections	Final Elements
P		....	1.562975 days
e	0.08	−0.029	0.051 ± .035
ω	119°	$+4^{\circ}.92$	$123.92 \pm 34^{\circ}.4$
K	30.5 km/sec.	−0.22 km/sec.	30.28 ± 1.38 km/sec.
T	J.D. 2422650.073	$+0^{d}.009$	$2422650.082 \pm 0^{d}.132$
γ	+71.7 km/sec.	−0.40 km/sec.	+71.3 km/sec.
$a \sin i$		....	650000 km
$\frac{m^3 \sin^3 i}{(m+m_1)^2}$		....	0.0045 ⊙

BOSS 2447

This was announced as a spectroscopic binary in a list published by Adams and Joy.[1] The single set of lines appearing on its spectrograms, which show that its class is G2 and its absolute magnitude +4.1, is quite satisfactory for measurement on properly exposed plates. Table IV gives the data for the twenty-eight plates which have been obtained. With $P = 19.4589$ days all observations arranged themselves satisfactorily within a single period and furnished the basis for the preliminary elements. Since the observations extend over an interval of approximately sixty revolutions of the star in its orbit, this period was taken as definitive. By the method of least squares the other five elements were corrected, equal weights being given to all of the twenty-seven plates which were used. Plate γ 6662, whose residual stood out as the only large one, was arbitrarily rejected from the least-squares solution, and Plate γ 9997 was obtained after the elements were derived. Data for both plates are given in Table IV, however. The quantity Σpv^2 derived from the elements, corrected as indicated by the solution given above, is 18 per cent less than for the preliminary elements, and substitution of the unknowns into the equations of condition gave residuals comparable with those derived from an ephemeris calculated with the corrected elements, which are therefore adopted as final. The preliminary elements, corrections, final elements, and probable errors are given in Table V. The probable error of a single observation of velocity of weight unity is ±1.74 km/sec. The barred circle in Figure 2 represents the velocity from Plate γ 9997.

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[1] *Publications of the Astronomical Society of the Pacific*, 31, 41, 1919.

TABLE IV

OBSERVATIONS OF BOSS 2447

Plate No.	Date	G.M.T.	Phase	Velocity	O−C
				km/sec.	km/sec.
γ 6396	1917 Nov. 27	$0^{\mathrm{h}}01^{\mathrm{m}}$	$9^{\mathrm{d}}.021$	+21.6	+ 1.5
6591	1918 Jan. 22	22 51	7.598	+29.1	+ 3.8
6662*	30	22 19	15.574	+ 4.3	+11.8
6706	Feb. 28	20 43	5.589	+29.0	− 1.4
6894	May 1	16 59	9.057	+20.2	+ 0.3
6943	30	16 43	18.589	− 2.8	− 0.3
C 346	1920 Apr. 3	15 02	11.456	+11.4	+ 1.8
352	4	18 43	12.609	+ 4.2	− 0.1
356	5	17 00	13.537	− 1.6	− 1.8
γ 9123	6	15 22	14.469	− 5.0	− 1.2
9130	7	17 04	15.540	− 2.5	+ 4.9
9179	May 6	15 39	5.563	+31.7	+ 1.2
9186	7	15 35	6.560	+30.4	+ 2.0
C 410	7	15 35	6.560	+29.3	+ 0.9
γ 9255	June 4	16 16	15.130	− 6.7	− 0.5
9266†	6	15 58	17.017	−10.1	− 1.2
9272	7	15 56	18.016	− 5.4	+ 0.6
9667	Oct. 26	0 52	2.813	+24.0	− 3.9
9688	29	1 03	5.821	+27.3	− 2.7
9767	Nov. 26	1 05	14.367	− 5.5	− 2.1
9774	27	0 53	15.359	− 8.6	− 1.8
9876	1921 Jan. 28	20 14	0.326	+ 8.2	− 0.1
9886	29	22 21	1.414	+17.9	− 0.8
9889	30	22 33	2.422	+29.8	+ 3.9
9927	Feb. 20	16 36	3.716	+31.4	+ 0.8
9935	21	20 56	4.896	+34.5	+ 3.3
C 896	22	17 10	5.739	+24.4	− 5.7
γ 9969	26	21 46	9.930	+11.9	− 4.4
9997‡	Mar. 16	17 06	8.277	+19.2	− 3.7

* Not used in least-squares solution.
† Poor plate.
‡ This plate obtained after elements were derived.

TABLE V

	Preliminary Elements	Corrections	Final Elements
P		....	19.4589 days
e	.200	+ .006	0.206 ± .002
ω	248°.5	+4°.00	252°.5 ± 7°.70
K	20.5 km/sec.	−0.29 km/sec.	20.21 ±0.49 km/sec.
T	J.D. 2422426.555	$+0^{d}.079$	$2422426.634 \pm 0^{d}.384$
γ	+12.49 km/sec.	−0.16 km/sec.	+12.3 km/sec.
$a \sin i$		....	5295900 km
$\frac{m^3 \sin^3 i}{(m+m_1)^2}$		....	0.0157 ⊙

Mount Wilson Observatory
October 1921

AN INVESTIGATION OF THE CONSTANCY IN WAVE-LENGTH OF THE ATMOSPHERIC AND SOLAR LINES[1]

By CHARLES E. ST. JOHN and HAROLD D. BABCOCK

ABSTRACT

Constancy in wave-length of atmospheric lines in the solar spectrum.—In 1915 Perot reported having found the wave-length of an O line considerably longer at noon than at sunrise and sunset. Since these lines are constantly used as standards of wave-length in solar observations, a study of the *wave-length as a function of the altitude of the sun* was made for a number of atmospheric lines in the B group at λ 6867, in the a group at λ 6276, and in the water-vapor band near λ 5900. Measurements of 25 plates show no indication of a variation greater than the accidental error. Moreover, spectrograms and data accumulated at Mount Wilson since 1911 give wave-lengths, some determined with reference to solar lines and some with reference to arc lines, which agree very closely, the difference from the general mean shown by any plate seldom exceeding 0.001 A. This negative result indicates the absence of high velocity *radial currents in the earth's atmosphere* and justifies the use of atmospheric lines as a reliable standard of reference even in work requiring the highest precision.

Constancy in wave-length of lines from the center of the solar disk.—While Evershed reported in 1919 remarkable variations amounting to several thousandths of an angstrom, the experience at Mount Wilson is that the more carefully the solar wave-lengths are compared with standard arc lines, the smaller the deviations from spectrogram to spectrogram become. Evidence from 13 plates is presented. These results prove that the *radial convection currents in the sun*, while not absent, are remarkably constant, apparently downward at high levels and upward, but small, at low levels.

In spectrographic observations on the solar spectrum for determining wave-lengths and displacements of the Fraunhofer lines with high precision, a desideratum is some means of procuring a simultaneous comparison spectrum under the same conditions of illumination as those obtaining for the spectrum under investigation. Provided their wave-lengths are constant, ideal conditions occur when atmospheric lines are used for reference in solar observations, since in the instrument the path of the light is identical for both classes of lines.

It has been generally assumed that under all practical conditions of solar observation the velocities of terrestrial atmospheric movements are of such an order that no measurable Doppler effect is produced. It would require motion in the line of sight of approximately 125 miles an hour to cause a displacement of 0.001 A at λ 6000. Some observations reported by Perot,[1] however, give very remarkable and surprising results, and if confirmed would show that the atmospheric lines are not constant in wave-length and hence not reliable as standards of reference. He measured with an interferometer the wave-length of an oxygen line in the B group at different hours of the day and concluded that the wave-length increased from morning to noon and decreased from noon to evening. He interprets his results as showing a recession of the absorbing centers from the surface of the earth with a radial velocity of about 3 km per second.

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[1] *Contributions from the Mount Wilson Observatory*, No. 223.

In view of the importance of the atmospheric lines in solar observations we have examined for change in wave-length during the day the atmospheric lines in three spectral regions, namely, in the B group at λ 6867, in the *a* group at λ 6276, and in the water-vapor band near λ 5900, and have supplemented these observations from spectrograms taken for other purposes through a series of years. For much of the measurement and reduction we are under obligation to Miss Ware, Miss Miller, and Miss Keener.

Oxygen lines in the B group.—On two occasions grating spectrograms of the center of the sun were taken in the first order of the 75-foot spectrograph of the 150-foot tower at intervals from sunrise to sunset. The wave-lengths of eight oxygen lines were obtained from solar lines whose wave-lengths were corrected for the earth's motions. The statistical results are given in Table I. The lines are identified by their Rowland wave-lengths in the first and fourth columns and in the other columns are given the deviations from the mean for the hours of observation. In Figure 1*A* the average deviations from the general mean are plotted against the corresponding altitudes of the sun.

From our investigation on solar wave-lengths in the international system we extract the data given in Table II. These measurements were made during June, 1919, with the interferometer attached to the Snow telescope. Our unpublished international values of solar lines were used as standards. The first column contains the preliminary wave-lengths in the international system of 28 oxygen lines in the B group. In the other columns are shown

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[1] *Comptes Rendus*, 160, 549, 1915.

TABLE I

Relative Wave-Lengths of Oxygen Lines of the B Group at Different Hours of the Day. Grating Spectrograms

(Unit for residuals = 0.001 A)

λ Rowland	February 25–26, 1919, P.S.T.			λ Rowland	June 5, 1919, P.S.T.						
	7:00	12:00	4:37		6:27	9:12	11:56	2:57	3:02	5:41	5:50
6879.288	+2	−3	0	6877.882	+4	0	−2	−2	0	0	0
6880.172	+6	+2	−7	6880.172	+4	+2	−2	−1	+1	−2	−2
6884.076	0	+2	−3	6884.076	+4	−4	−5	0	+1	+1	0
6886.000	+1	+2	−3	6900.199	+2	−4	−3	+2	+2	−2	+4
6913.448	−7	+5	+1	6905.271	+2	−2	+1	−1	0	−3	0
6914.337	−5	0	+6	6909.676	+1	0	0	−3	+1	−1	−1
6918.370	....	+1	−1	6913.448	+2	0	−1	−3	−1	0	−4
6919.250	....	−1	+2	6918.370	+1	0	+3	0	+2	−2	−1
Mean	−0.5	+1.0	−0.6	Mean	+2.5	−1.0	−1.1	−1.0	+0.6	−1.1	−0.5
Altitude	7°00′	46°40′	10°20′	Altitude	17°40′	51°30′	78°10′	49°40′	48°40′	16°10′	14°20′

the deviations from the mean values for altitudes varying from 12°50′ to 75°40′. The observed mean wave-length at the highest altitude is 0.0012 A less than the general mean.

Oxygen lines of the a group.—On June 7 and 9, 1919, spectrograms of the center of the sun were taken with the 75-foot spectrograph at intervals from 6:00 A.M. to 6:00 P.M. Eight oxygen lines of the *a* group were measured in terms of solar standards. After correction of the reference lines for the earth's motions, the mean wave-length at noon exceeded the mean for the two lowest altitudes by 0.0016 A. The numerical data are given in Table III; the relative results are plotted in Figure 1*B* against the hours of observation.

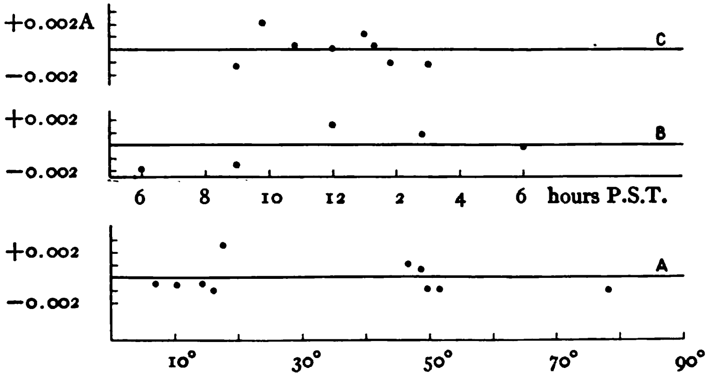

FIG. 1.—*A*. Deviations of oxygen lines in B group from mean wave-length; for different altitudes of sun.

B. Deviations of oxygen lines in *a* group from mean wave-length; at different hours of the day. Solar standards.

C. Same as *B*, referred to absorption lines of iodine as standards.

In order to eliminate solar standards and obtain an independent determination, spectrograms of sunlight filtered through iodine vapor were taken with an interferometer using two absorption lines of iodine as the fixed references. These observations were made in the Pasadena laboratory in April, 1919, with an etalon of 20 mm separation giving a mean order of interference of 63400 for the eight measured oxygen lines of the *a* group. The deviations from the mean wave-length at the hours of observation are plotted in Figure 1*C*. The wave-length at noon shows no deviation from the mean of the series.

TABLE II

Relative Wave-Lengths of Oxygen Lines of the B Group at Different Hours of the Day. Interferometer Spectrograms

(Unit for residuals = 0.001 A)

June 6, 15, 16, 1919, P.S.T.

λ I.A.	June 6 9:08	June 6 10:32	June 6 12:36	June 16 3:40	June 15 5:11	June 16 6:03
6872.265	−3	−1	+1	+2	+3	−3
6872.861	+1	−3	−3	+3	0	....
6873.816	−2	0	0	0	+4	−2
6874.671	−2	+1	−3	−2	+5	0
6875.608	0	0	0	0	+3	−5
6876.733	−2	+3	0	−2	−2	+1
6877.655	+2	+2	−2	−3	+1	−1
6879.058	+2	+3	−3	+2	−1	−10
6879.947	+4	0	−3	−1	−1	−5
6883.850	+1	+1	0	−1	+1	−6
6885.772	−1	+1	−1	−1	+4	−4
6886.761	−1	+1	−2	0	+3	−3
6888.967	+1	+1	−1	0	0	−3
6889.921	−2	0	0	+2	−3	....
6892.389	−6	+3	−3	+2	−1	....
6893.328	0	0	−5	−2	+6	−4
6896.056	0	+2	+2	−2	−3	−2
6896.984	+1	0	0	+1	−3	−3
6899.972	−1	0	−1	+2	−1	+2
6900.887	−2	+1	−4	+0	+4	−5
6904.137	−1	+2	−2	−2	0	....
6905.137	−1	+2	−2	−2	0	....
6908.553	+1	+1	−3	−1	+3	−2
6909.450	−1	+2	−2	−1	+1	−3
6913.219	0	+3	+1	−4	+1	....
6914.109	0	+5	0	−4	+1	....
6918.140	−2	+2	−2	+1	−3	....
6919.022	+1	0	−1	−1	+3	....
Mean	−0.4	+1.1	−1.4	−0.5	+1.0	−2.9
Altitude	50°40′	67°20′	75°40′	41°10′	22°30′	12°50′

From spectrograms and data accumulated since 1911, when the *a* band was first studied at Mount Wilson, a yearly record is available. The results for eight lines are given in Table IV. The mean wave-lengths for the eleven-year period are in the first column and those for the separate years in the succeeding columns. The mean wave-length of the group is shown at the bottom of the table. The spectrograms were taken in various months, on various days, and at various hours. It would seem that frequently occurring disturbances should have been detected; but on none of the scores of spectrograms measured is there definite evidence of radial movements of the absorbing regions of the atmosphere. The difference from the general mean shown by any plate seldom exceeds 0.001 A.[1]

TABLE III

WAVE-LENGTHS OF OXYGEN LINES OF THE a GROUP AT DIFFERENT HOURS OF THE DAY. GRATING SPECTROGRAMS

June 7–9, 1919, P.S.T.

λ I.A.	June 7 6:00	June 7 9:00	June 7–9 12:00	June 9 2:50	June 9 6:00
6280.404..........	.403	.402	.406	.404	.403
6281.187..........	.185	.186	.189	.189	.186
6281.964..........	.962	.962	.966	.966	.963
6290.230..........	.229	.230	.232	.230	.229
6295.187..........	.185	.184	.188	.187	.188
6295.969..........	.967	.968	.970	.971	.968
6302.009..........	.006	.006	.010	.009	.010
6302.770..........	.768	.769	.772	770	.771
Mean .215........	.2131	.2134	.2166	.2158	.2148
No. of plates......	2	1	3	2	4
Altitude..........	12°30′	49°10′	78°40′	51°20′	12°40′

Water-vapor lines near λ 5900.—To complete the investigation on atmospheric lines, grating spectrograms of the center of the sun's disk were taken at high and low sun in the region of the rain-band near D_1 and D_2. The results of the measures of eleven lines on twenty-four spectrograms are shown in Table V. Solar lines were used as standards. The mean wave-lengths in the international system are in the first column, weighted according to the number of exposures. For the eleven lines the wave-length at high sun exceeds the mean at low sun by 0.001 A.

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[1] This confirms and extends in time other observations on the a group. Royds, *Annual Report, Kodaikanal Observatory*, 1917; St. John and Babcock, *Publications of the Astronomical Society of the Pacific*, 31, 178, 1919.

TABLE IV

YEARLY RECORD FOR ATMOSPHERIC LINES OF THE α GROUP

MEAN WAVE-LENGTHS I.A. UNITS	STANDARDS OF REFERENCE										
	Solar Lines Corrected for the Earth's Motions								IRON ARC LINES USED AS STANDARDS		
	At Center			At Limb, Corrected for Limb-Center Shift							
	1911	1912	1913	1914	1915	1916	1917	1918	1919	1920	1921
6280.400	.400	.400	.403	.403	.400	.397	.400	.402	.402	.398	.403
6281.186	.183	.185	.185	.186	.184	.183	.185	.185	.186	.187	.188
6281.963	.961	.960	.963	.964	.966	.962	.963	.963	.963	.963	.964
6290.229	.230	.229	.228	.230	.229	.228	.226	.229	.229	.229	.229
6295.186	.187	.183	.184	.186	.185	.184	.187	.182	.187	.185	.185
6295.968	.970	.967	.968	.969	.968	.967	.967	.965	.968	.968	.970
6302.008	.009	.007	.010	.012	.006	.008	.005	.005	.008	.007	.009
6302.771	.771	.771	.769	.776	.772	.768	.771	.767	.770	.772	.772
Mean .214	.214	.213	.214	.216	.214	.212	.213	.212	.214	.214	.215
No. of plates	8	2	2	3	4	2	2	2	22	13	2

TABLE V

WAVE-LENGTHS OF WATER-VAPOR LINES AT DIFFERENT HOURS OF THE DAY

(March and June 1921)

λ I.A.	P.S.T. of Observation			
	7:03	11:40	3:16	5:40
5885.982	.980	.983	.984	.980
5887.227	.226	.227	.230	.226
5887.665	.664	.668	.663	.664
5892.402	.396	.405	.403	.402
5909.001	.998	.000	.004	.002
5913.999	.998	.996	.004	.999
5919.058	.056	.059	.059	.057
5919.646	.644	.647	.646	.646
5924.276	.275	.275	.278	.276
5932.097	.096	.097	.098	.097
5932.788	.788	.788	.788	.788
Mean .4675	.466	.468	.469	.467
Altitude	16°	61°	35°	13°

When the observations on atmospheric lines at high sun are compared with the mean for all altitudes, the residuals, high sun *minus* mean, are as follows:

Source	Lines Observed	Altitude High Sun	Residuals
Table I	B group	78°10′	−0.0010 A
Table II	B group	75 40	−0.0012
Table III	α group	78 30	+0.0016
Table IV	Water-vapor	61 20	+0.0005
Mean			0.0000

We are unable to account for the wide divergence between our measures and those of Perot. Within the error of measurement our series of observations show no variation in the wave-lengths of the terrestrial lines with the altitude of the sun. It will be recalled that Dunér,[1] in his determination of solar rotation in 1887–1889, used the terrestrial line λ6302.209 as a fixed point of reference, and in 1899–1901 the other component of the pair, λ6302.975. If the terrestrial atmosphere in the absorbing region is subject to radial velocities of 3 km per second his consistent values are remarkable, particularly for the highest latitude where the observed rotational velocity was only 0.39 km per second. It must be admitted that radial velocities in the earth's atmosphere of the order of 3 km per second, if they do occur, are extremely rare and very local. It seems to us more probable that a different interpretation should be sought for Perot's observations. His interpretation requires for the absorbing centers a radial velocity of 6700 miles per hour. That such enormous radial velocities occur even in attenuated regions of the earth's atmosphere is open to question, and it is more doubtful still that the radial component would remain constant for a month over a given locality. From our observations we feel justified in using the atmospheric lines as reliable standards of reference, even in work requiring the highest precision.

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[1] *Über die Rotation der Sonne*, Upsala, 1906.

On constancy of wave-length of solar lines.—Jewell[1] was the first to raise the question of variability in the wave-lengths of the solar lines. He says,

> Another effect of this investigation may be to make the lines of the solar spectrum step down from the commanding position which they have occupied, as standards of reference.

At the period of Jewell's investigation the instability of many lines in the arc spectra of metals had not been recognized. Moreover, the records of the spectrograms taken by Rowland and used by Jewell were not sufficiently detailed to determine corrections depending upon the earth's motions. Under these conditions the conclusion reached by Jewell loses much of its weight. Evershed[2] reports that

> a continuous series of sunlight and Fe spectra was taken to test the constancy of the sun-arc displacement. Confining attention to the region 4337–4531 and to lines not subject to pole-effect in the arc, it was found that some remarkable variations occurred amounting to several thousandths of an angstrom. The variations are of two kinds: a general change for all the lines in the region studied, and a change affecting particular lines or groups of lines.

He says,

> These displacements may be observed at the center of the disk, but up to the present they have not been found very near the limb. It appears therefore that, unlike the displacements in the penumbra of spots, they may be

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[1] *Astrophysical Journal*, 3, 113, 1896.

[2] *Annual Report, Director, Kodaikanal and Madras Observatories*, 1919, p. 2.

due to movements normal to the surface or having a component normal to the surface.

At Mount Wilson it has been found that in the determination of solar wave-lengths from arc standards, the more carefully the iron arc is controlled, the more completely instrumental displacements are eliminated, the more accurately the image of the sun is centered,[1] and the more exactly the axes of the light-cones incident upon the grating are made to coincide, the smaller the deviations from spectrogram to spectrogram become.

A comparison of four observations with the grating spectrograph from April to June, 1921, and the mean of measures in 1917 for the 13 solar lines common to all plates gives for the fractional part of the mean wave-length the following:

		1917, Mean	1921, April 21	1921 April 29	1921 May 1	1921 June 1
13 lines..........	λ 4337–λ 4401	0.5545	0.5552	0.5550	0.5565	0.5554

The three April-June observations for the other common lines also show good agreement in the fractional part.

		1921, April 21	April 29	May 1
17 lines...............	λ 4489–λ 4607	0.5574	0.5594	0.5593
15 lines...............	λ 4615–λ 4688	0.3942	0.3941	0.3945
20 lines...............	λ 4707–λ 4784	0.5217	0.5217	0.5216

These wave-lengths are determined from the arc standards, not by measuring the displacements between the center of the sun and arc, but by using the arc lines as standards of reference. Five interferometer spectrograms likewise referred to arc standards, taken on August 5, 1920, at intervals from 10:44 A.M. to 5:03 P.M. P.S.T. and one taken a year later on August 8, 1921, give deviations from the mean of the six as follows for a group of 38 solar lines:

		August 5, 1920					August 8, 1921
		10:44	2:34	3:30	4:22	5:03	4:50
38 lines...	λ 5497–λ 5862	.0000	+.0011	+.0003	−.0020	+.0005	.0000 A

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[1] *Observatory*, 43, 260, 1920.

The degree of constancy of the solar wave-lengths at the center of the sun shown by these determinations is further evidenced by considering the data in Table IV from this point of view.

The wave-lengths of the oxygen lines in the table, except for the years 1920–1921, were referred to solar standards determined as above. In 1911, 1912, 1913, and 1919 the wave-lengths of the solar lines at the center of the disk were used. For the years 1920–1921 the oxygen lines were referred directly to the iron-arc standards. This completes the cycle: arc–sun–atmospheric lines–arc. The closing of the cycle is practically perfect. Unless it is assumed that continuously compensating changes in the wave-lengths of the solar and terrestrial lines occur, the agreement between the first and last sections of Table IV may be taken as a measure of the constancy in wave-length of the atmospheric lines and of the solar lines at the center of the disk, and also of the accuracy of their determination in terms of the international iron-arc standards. In 1914–1918 the results are less dependable, as the measures were made upon spectrograms of the limb and it was necessary to apply a correction for the limb-center shift, a somewhat uncertain quantity for a single observation. From an extended series the mean limb-center shift for solar lines in this spectral region is 0.007 A with a range of 0.002 to 0.010 A.

In view of the probable existence of radial convection currents on the sun the relative constancy of wave-length at the sun's center is somewhat surprising. At any given level, the radial velocity appears to reach a comparatively steady state, differing from level to level and in widely different levels even opposite in direction. For the strong lines—high-level lines—the motion is apparently downward. The great intensity of these lines at high levels is probably dependent on the increased ionization under the low pressures at the high elevations.[1] As yet too little is known of the ionizing potentials of the elements involved for a definite conclusion. For very low-level lines the motion is small but upward. At both low and high levels we appear, however, to be dealing with remarkably steady states. In this regard there is a marked difference between the center and the limb, the range of deviation from

[1] Megh Nad Saha, *Philosophical Magazine*, 40, 472, 1920.

plate to plate at the limb being four- to five-fold that at the center. Though the causes of the motions in the solar and terrestrial atmospheres are different, there may be common characters in the general atmospheric movements due to the similar damping influences. In both cases centrifugal velocities are opposed by gravitation, very powerfully so in the sun, while tangential movements are damped by the smaller forces of friction. Thus through a kind of selective action the tangential components would tend to persist and the radial components to be quenched, so that in both the solar and terrestrial atmospheres the prevalent atmospheric flows would tend to be tangential. In the sun the opposing effects of light-pressure and gravitation may have influence in producing the steady state at any given level, the different levels of the absorbing centers being again a selective result as the equilibrium between light-pressure and gravitation depends upon the ratio between the mass and diameter of the centers.

From these observations it is evident that differences between solar wave-lengths at the sun's center and in terrestrial sources may be determined with an accuracy comparable to that for the terrestrial lines. Upon the magnitude of these differences, their variation over the solar disk, from element to element, from line to line, and from wave-length to wave-length, rests the possibility of disentangling the causes of the displacement of solar lines, a formidable but far from hopeless undertaking.

Mount Wilson Observatory
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THE WAVE-LENGTH IN ASTRONOMICAL INTERFEROMETER MEASUREMENTS[1]

By J. A. ANDERSON

ABSTRACT

Effective wave-length in measurements with astronomical interferometer.—The separation of double stars and the diameters of stellar disks are measured by the interferometer in terms of an effective wave-length λ which is such that if the stellar object emitted monochromatic light of wave-length λ the setting of the interferometer would be that actually found. The accuracy of setting obtainable in these measurements is such that in each case λ should be known to a few tenths of 1 per cent. *For any star*, the normal effective wave-length may be computed from the effective temperature corresponding to its spectral type, by combining the Planck radiation law with the mean transmission curve for the atmosphere and with the A.I.S. visibility-curve for the normal eye. The corrections required for variations of atmospheric absorption and of eye sensitivity from the assumed values may best be obtained experimentally by observations on sunlight, since in any case the correction is equal to the difference between the observed and computed effective wave-lengths for sunlight under the same conditions. The method and apparatus used in the *determination of the effective wave-length for sunlight* are described, and the results of some observations on the *variation of* λ *with zenith distance and atmospheric conditions* are given. On Mount Wilson the values found vary from 5660 A at sunrise and sunset to 5510 A for zenith distances up to 60°; at Pasadena the variation with zenith angle was more rapid. The wave-length was greater on cloudy days and least after a rain.

Astronomical interferometer.—The distribution of intensity in the interference patterns, for both circular and rectangular apertures, is fully discussed, and in the particular case of a double star is given mathematically for all points of the focal plane, for all orientations of the apertures. *The effect of the size of the apertures on the setting* is nil when rotation is used and also when the setting is made for minimum visibility of the fringes; but when the apertures are separated until minimum intensity at the center is obtained, it is found both theoretically and experimentally that for apertures of width a and separation D, the effective separation is very closely equal to $D/[1+K\ (a/D)^2]$ for D/a greater than 3.5, where K is 0.223 instead of 0.765 as given by Hamy.

The interferometer as used in astronomical measurements consists of two apertures alike in shape and size, placed either in front of the objective or in the converging beam a short distance in front of the focal plane. The idea of using a telescope in this manner appears to have originated with Fizeau,[2] and the first actual trials were made by Stephan.[3] Michelson[4] carried both the theory and applications of the method considerably farther than his predecessors; he gave an analytical treatment for the case of a pair of narrow slits, and applied it to the determination of the diameter of Jupiter's satellites. Hamy[1] treated the case of narrow slits, and also that of rectangular apertures whose width cannot be neglected in comparison to their separation, and applied his results to a measurement of the diameter of Jupiter's satellites and of some of the brighter asteroids. Comstock[2] used the method for a determination of the effective wave-length of star light. The work of the past two years is so recent that references to it need not be given here in detail.

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[1] *Contributions from the Mount Wilson Observatory*, No. 222.

[2] *Comptes Rendus*, **66**, 934, 1868.

[3] *Ibid.*, **78**, 1008, 1874.

[4] *Philosophical Magazine*, **30**, 1, 1890.

The purpose of the present paper is to call attention to the necessity of knowing quite accurately the effective wave-length of the source which is studied; to describe and illustrate a method for determining this quantity quickly and accurately; and to discuss in some detail the effect of using apertures of different shapes and sizes.

Consider first the determination of the diameter of a star, assumed to appear as a uniformly luminous circular disk. If the apertures are narrow in comparison with their separation, the angular diameter is given by

$$\beta=\frac{1.22\lambda}{D} \tag{1}$$

where β is the angular diameter, D the distance between the centers of the apertures corresponding to the first vanishing of the interference fringes as the distance is increased, and λ the wave-length of light. The two quantities which determine β are λ and D. Experience indicates that the probable error in the determination of D under good conditions is of the order of 1 per cent; and accordingly if the uncertainty of the true value of λ exceeds a few tenths of 1 per cent, the probable error of β will be larger than that of D.

Again, consider a double star whose angular separation is to be measured. This is given by

$$\alpha=\frac{\lambda}{2D} \tag{2}$$

where α is the angular separation, the other quantities being the same as in (1). In this case, under good conditions, the uncertainty in the value of D is considerably less than 1 per cent.

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[1] *Bulletin Astronomique*, 16, 257, 1899.

[2] *Astrophysical Journal*, 5, 26, 1897.

Other cases might be given, but always the quantity to be measured will be expressed in the form $K\lambda$, where K is determined by the actual observation, while λ in general must be found by some other means, and since λ is always a simple multiplier, it is necessary or at least desirable to know its value with a probable error considerably smaller than that involved in K.

At present all observations with the interferometer are made visually, and hence the value of λ will fall somewhere near the middle of the visibility curve of the eye, at least for sources emitting white light; or, we may say that λ will have a value lying somewhere within the range 5400 A to 6000 A. It follows from what has just been said that, if possible, λ should be known for a given source within 10 A to 20 A.

Definition of the effective visual wave-length.—*a*) Let an artificial double star be illuminated with the white light whose effective visual wave-length is required; let this double star be viewed with an interferometer, and the latter adjusted so that the interference fringes just disappear according to equation (2). Leaving the interferometer unchanged we now illuminate the double star with the monochromatic light whose known wave-length can be varied at will. The value of λ which just makes the fringes disappear is defined as the effective visual wave-length of the white source in question.

b) Analytically[1] the definition may be given as follows: Take rectangular co-ordinates in the focal plane with the origin on the axis of the telescope. Let the geometrical images of the components of a double star fall at $(+c, 0)$ and $(-c, 0)$. Let the apertures be narrow slits separated by a distance D, and the focal length of the objective be F.

If $I(\lambda)d\lambda$ be the visual effect of the radiation of the source lying between the limits λ and $\lambda+d\lambda$, the intensity along the x-axis is given by

$$J=\int_0^\infty I(\lambda)d\lambda\left[\cos^2\frac{\pi D(x-c)}{F\lambda}+\cos^2\frac{\pi D(x+c)}{F\lambda}\right] \qquad (3)$$

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[1] This statement was first suggested to the writer in a letter from Dr. C. M. Sparrow.

When D is very small, J has a maximum at $x=0$. As D increases, J diminishes for $x=0$, reaching a value of zero when D is somewhere near $F\lambda_0/2c$ where λ_0 is an arbitrary value of λ near the middle of the range for which $I(\lambda)$ has an appreciable value. Both for D small and for D having a value near $F\lambda_0/2c$, J oscillates as x is varied; but for a value of D in the neighborhood of $F\lambda_0/4c$, these oscillations become very small, and in particular at $x=0$, $J(x)$ will have a stationary value for some particular value of D. The condition for this is

$$\frac{\partial^2 J}{\partial D^2}=0;\ \text{for } x=0 \qquad (4)$$

Solving (4) we have $D=F\lambda_0'/4c$, where λ_0' is the effective visual wave-length required.

For a source such as a given star, $I(\lambda)$ is defined as follows: Let $E(\lambda)$ represent the energy-curve of the star, as determined outside our atmosphere; let $T(\lambda)$ be the transmission-curve of the atmosphere at the time and place of the observation; and let $V(\lambda)$ be the visibility-curve of the eye of the observer; then

$$I(\lambda)=E(\lambda)T(\lambda)V(\lambda).$$

Some questions now naturally arise, among which the following are important enough to state:

1. If in (*a*), instead of using an artificial double star, we use an artificial star disk, would the same value of λ be obtained?

2. If the components of a double star are of different spectral types, or, what amounts to the same thing, if the effective wave-length of the components is not the same, what wave-length is to be used in calculating the angular separation from the data given by an observation with the interferometer?

3. In case of a star disk darkened toward the limb, if this darkening is accompanied by a color change, what value of λ is to be used?

It is important to remember that when the observations are made visually, different observers will not in general find the same effective wave-length for a given source. In (*b*) above this is provided for by the factor $V(\lambda)$, which may differ slightly for different observers. In (*a*) it is automatically provided for by reason of the fact that the same observer is supposed to make both the adjustments. Again, since $T(\lambda)$ enters as a factor in $I(\lambda)$, an observer will not obtain the same value of λ for a given source under different conditions of observations as to time and place.

Only a beginning has thus far been made in the application of the interferometer to astronomical measurements, and it is too early to predict what its future may be; yet it will undoubtedly be valuable in certain restricted fields; for example, in the direct determination of star diameters and in the measurement of very close double stars of particular types. No doubt it will be applied successfully also in other fields. Now, as has already been pointed out, the result of any interferometer measure[1] will be expressed in the form

$$\theta = K\lambda \tag{5}$$

where K is given by the observation, and λ depends upon the object observed. If λ did not depend upon local conditions such as the observer, the time and place of the observation, and the instruments employed, the reduction would be a relatively simple matter; for this case the value of λ could be found, once and for all, for stars of all spectral types. It is clear, however, from what has already been said, that this may involve errors very much larger than the errors of observation. On the other hand, it will obviously be quite impracticable for each observer to determine independently the value of λ for every observation he makes. The following plan will, it is hoped, remove most of the difficulties. We will leave out of account objects such as the planetary nebulae, that is, objects having discontinuous spectra, for in such cases there is no difficulty in determining the value of λ with sufficient accuracy. Consider then only objects like the great majority of the stars, having continuous spectra, more or less similar to spectra of black bodies at various temperatures. The fact that no star radiates exactly like a black body is of little importance, because the black-body temperature will be used merely as an auxilliary in the general scheme. The black-body temperature of stars has been determined for all or nearly all the brighter stars, and these include nearly all of the spectral types that it will be necessary to consider.

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[1] The position-angle of a double star is an exception.

Using Planck's law, the mean transmission-curve of the atmosphere, and the adopted *A.I.S.* visibility-curve of radiation, we may compute by equation (4) the value of λ_0' for a few temperatures from 3000° to 20,000° absolute. A curve passed through the computed points will enable one to read off λ_0' for any temperature within the range. The absolute value of the wave-lengths thus derived may not be correct, but this does not matter, for their relative values will be sensibly right. To illustrate the use of such a curve in practice, suppose that an observer has made an interferometer measure of a star of spectral type F0; assume also that he has made a determination of the effective wave-length of sunlight by the method given later in this paper. From tables giving the effective temperatures of stars of different spectral types, he reads off the temperature for an F0 star and for a star of the same spectral type as the sun. Let these temperatures be, respectively, T_1 and T_2. From the curve the corresponding wave-lengths are λ_1 and λ_2. If now the measured wave-length of sunlight is λ_0, the proper wave-length for the F0 star will be

$$\lambda = \lambda_0 + \lambda_1 - \lambda_2 \tag{6}$$

Of course, as the determination of λ_0 is necessarily made in daytime while the observation of the star is made at night, it is probable that the atmospheric conditions will have changed in the interval. But as different determinations of λ_0 will soon show about what correction is required for variations in observing conditions, this difficulty is not very serious.

Experimental determination of effective wave-length.—In Figure 1 let *A* be an artificial double star, consisting of two small pinholes whose linear separation is δ, illuminated by white light such as sunlight. At *B* are two circular apertures separated by the distance *D*, so mounted that they can be rotated about the axis of the telescope *C*. Observations are made through the compound

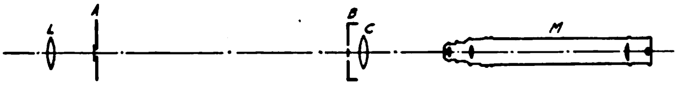

FIG. 1

microscope *M*. The distance *A* to *B* will be denoted by *L*. The angular separation of the pinholes as viewed from *B* is δ/L. The value of *D* for the first disappearance of the interference fringes, if the line joining the pinholes is parallel to that joining the apertures at *B*, is, by equation (2),

$$\frac{\delta}{L}=\frac{\lambda}{2D}$$

or, if the two lines make an angle θ with each other,

$$\frac{\delta}{L}=\frac{\lambda}{2D\cos\theta} \tag{7}$$

All the quantities in the equation except λ are directly measurable, and hence λ can be determined.

In order to form some idea of the accuracy to be expected in this determination, we will now examine in detail the measurement of each of the quantities entering into equation (7).

In the present experiments δ was of the order of 0.6 mm. The two pinholes were made in tinfoil, as follows: A fine needle was selected and mounted in a mechanics' pin-chuck. A small piece of tinfoil was smoothed out on a flat, smooth piece of soft metal such as brass or aluminum. A small round hole was made by resting the needle point on the tinfoil, the needle and chuck being vertical, and then carefully turning the chuck a few times. By always taking care to have the pressure about the same, holes of very nearly the same diameter may be made, and if care is used in rotating the chuck and in lifting the needle after the operation, the holes will be quite perfectly round. The diameter of the pinholes used was between 0.035 and 0.050 mm. Under a microscope the distance between such a pair of openings can be determined with a probable error of about 0.0002 mm. Since δ is about 0.6 mm, this corresponds to a probable error of the order of 1 in 3000. Before measurement the tinfoil is mounted on a metal plate through which a hole about 2 mm in diameter has been drilled, the apertures in the tinfoil being directly over this hole.

Two sources of error are to be noted in using such an arrangement as an artificial double star:

1. The plane of the apertures will not in general be exactly perpendicular to the line of sight from the observing telescope, but will make an angle ϕ with this line. The apparent separation will then be not δ but $\delta \sin \phi$. With ordinary care in mounting the tinfoil and adjusting the plate, ϕ should not differ from 90° more than a degree, which would at most cause an error of 15 parts in 100,000, which is negligible.

2. When an image of the sun is formed on the apertures, the tinfoil becomes heated somewhat, causing δ to increase. The rise in temperature to be expected may be roughly estimated. The image-forming lens has an area of 16 cm^2, and its focal length was 1 meter, so the diameter of the image was roughly 1 cm. Of this $\frac{1}{25}$ fell on the tinfoil, which is equal to the amount of sunlight falling on $\frac{16}{25}$ cm^2, or 1 calorie per minute. Assuming that the tinfoil reflects 80 per cent of this, we have $\frac{1}{5}$ calorie per minute taken up by the tinfoil or 0.003 calorie per second. When equilibrium is established, the greater part of this energy passes by conduction from the tinfoil to the brass plate, which is so large that its temperature never rose appreciably above room temperature. A temperature gradient of 3° per centimeter in the tinfoil would more than suffice to carry this amount of energy to the brass plate—and hence we may conclude that, within a very few degrees, the tinfoil was always at room temperature, and that the error due to thermal expansion was less than 1 in 10,000, which is negligible.

The value of D was usually between 6 and 7 mm. The apertures were round holes made with a drill in a brass plate; they were always sensibly equal in diameter, and the determination of their distance apart could always be made with an error less than 1 part in 5000. Various diameters were used, from about 1 mm up to 2.5 mm. The effect of using apertures of different sizes will be discussed more fully below.

The value of L was about 900 centimeters, and with a good steel tape this distance could always be measured to a millimeter.

It follows that, as far as the quantities δ, D, and L are concerned, the sum of the errors made in their evaluation should never affect the result by as much as 1 part in 1000; and, since the wave-length of the light used was near 5500 A, the uncertainty arising from the instrumental constants is not more than 5 A.

It remains now to discuss the value of θ, which is the only quantity directly measured in an actual experiment. In order to make the discussion of θ intelligible it is necessary to describe and explain the appearance of the diffraction pattern and its system of interference fringes in some detail.

Consider the distribution of intensity in the focal plane, due to a single distant point source, when the objective is covered by a screen perforated by a single circular aperture of radius r. Denote the focal length by F, and let the origin of rectangular co-ordinates coincide with the optical axis. The intensity is given by

$$I=K\frac{4J_1^2\dfrac{2\pi r\sqrt{x^2+y^2}}{F\lambda}}{\left(\dfrac{2\pi r\sqrt{x^2+y^2}}{F\lambda}\right)^2} \tag{8}$$

where J_1 is a Bessel function of the first order.

This well-known expression states that at $x=0$, $y=0$, the intensity is a maximum and equal to K; as we move away from the origin, the intensity diminishes at first very slowly, then more rapidly, and reaches a value zero, when the argument of J_1 is equal to about 219°33′. For larger values of the argument the intensity increases, reaching a second maximum whose value is $0.0175\,K$ near 292°, and a second zero value near 401°, etc. This is, of course, simply the ordinary diffraction pattern of a circular aperture, or the so-called spurious star disk. In ordinary telescopic observations with relatively large apertures the entire disk and diffraction rings are apparently very small; but in interferometer work, on account of the small apertures and the very high magnification employed, the pattern may cover the larger part of the field of view.

The appearance is independent of the location of the aperture, as long as this falls entirely within the objective. Hence we will assume that it is placed so that its center is at a distance $D/2$ from the axis. Now let an equal aperture be uncovered, at the same distance from, but on the other side of, the axis. This second aperture alone would give exactly the same intensity as the first, namely that expressed by equation (8); but if we use both apertures together, the result will be the sum, not of the intensities as given by (8), but of the amplitudes formed by taking the square root of the right-hand side of (8), with proper attention to the relative phases of the two disturbances.

Let the apertures be so placed that the line joining their centers is parallel to the x-axis; in this case the amplitudes and phases will be equal at $x=1$. At $x=\pm F\lambda/2D$ the phases will differ by 180°, and since the amplitudes are equal, the resultant amplitude is zero. At $x=\pm F\lambda/D$, the phases differ by 360°, that is, they are in phase, and the amplitude will be the sum of the two, and so on.

It follows that the resultant intensity is now given by

$$I=I_0\frac{4J_1^2\dfrac{2\pi r\sqrt{x^2+y^2}}{F\lambda}}{\left(\dfrac{2\pi r\sqrt{x^2+y^2}}{F\lambda}\right)^2}\cos^2\frac{\pi Dx}{F\lambda} \tag{9}$$

I_0 being the intensity at the origin.

Hence the general outline of the pattern is the same as that given by a single aperture; we still have the central disk surrounded by diffraction rings; but the effect of using two apertures is to break up the disk and rings into a series of equally spaced interference fringes parallel to the y-axis—that is, perpendicular to the line joining the apertures. Along a line parallel to the y-axis the distribution of relative intensity is exactly the same as that in the pattern given by a single aperture; parallel to the x-axis we have this intensity multiplied by the factor $\cos^2 \pi Dx/F\lambda$.

Figure 2 gives a graphical representation of the intensity along the x-axis. The dotted curve is the intensity due to a single aperture, multiplied by the factor 4.

If we retain the apertures, but use two point-sources separated by an angular distance α, there will be two such patterns, whose centers are separated by the distance $2c=F\alpha$. If the centers fall at $(-c, 0)$ and $(+c, 0)$, respectively, the expression for the intensity becomes

$$I=I_0\left\{\frac{4J_1^2\left(\frac{2\pi r\sqrt{(x-c)^2+y^2}}{F\lambda}\right)}{\left(\frac{2\pi r\sqrt{(x-c)^2+y^2}}{F\lambda}\right)^2}\cos^2\frac{\pi D(x-c)}{F\lambda}\right.$$
$$\left.+\frac{4J_1^2\left(\frac{2\pi r\sqrt{(x+c)^2+y^2}}{F\lambda}\right)}{\left(\frac{2\pi r\sqrt{(x+c)^2+y^2}}{F\lambda}\right)^2}\cos^2\frac{\pi D(x+c)}{F\lambda}\right\} \qquad (10)$$

If the line joining the apertures makes an angle θ with the x-axis, then, letting $m=\tan\theta$, the expression for the intensity will be that given by equation (10), except that we write $x+my$ instead of x in the cosine factors. Figure 3 is a sketch showing the pattern for such a double point-source, the line joining the apertures making the angle θ with x-axis. The centers of the patterns are at P and P' respectively; the pattern centered at P' is shown in dotted lines. The straight lines represent the centers of the bright fringes, AA being the central fringe for P', $A'A'$ that for P'. The angle θ has been so chosen that the fringes due to P' are just midway between those due to P. If θ is diminished, the similarly lettered fringes will approach each other and the visibility will be increased; and similarly, if θ is increased, AA will approach $B'B'$, etc., and the visibility will again increase. The position represented is that of minimum visibility.

Remembering that the intensity in a pattern diminishes from the center outward, it is clear that at all points on the y-axis the intensity of one pattern is the same as that of the other, except for the changes due to the interference, and since the maxima in one pattern fall on the minima of the other, the result is very nearly uniform intensity along this line. To the right of the y-axis the intensity of P' is greater than that of P, and hence the maxima of the P' pattern become more and more prominent as we go to the right. To the left of the y-axis the P pattern is stronger and hence its maxima become increasingly prominent in this direction. The observer will therefore see two sets of fringes meeting just out of step near and along the y-axis. (a) If θ is decreased a certain amount, so that the distance between AA and $A'A'$ becomes smaller than the adjacent intervals, the corresponding bright fringe will start at A' on the right and run in a sensibly straight line to A on the left, crossing the y-axis exactly at the origin. (b) If θ is increased, we shall have a dark fringe running from A on the right in a nearly straight line to A' on the left, also crossing the y-axis at the origin. The change in θ necessary to pass from condition (a) to condition (b) depends upon the number of fringes in the pattern as here represented, being the smaller, the larger the number of fringes. For a case such as that in Figure 3, with 6 fringes across the central disk, the total required change in θ is about 5°. With 12 to 15 fringes it will be from 2° to 3°. It is evident, therefore, that a setting for the exact bisection can be made quite accurately—in general to a few tenths of a degree.

We can now estimate the probable error in the determination of θ. In one rotation of the aperture plate there are four values of θ which give the appearance just described, and a "set of observations" in the present experiments has consisted of five complete rotations. As a typical example we may take the measures made on Mount Wilson on June 18, 1921. Fourteen sets of observations were made of which only eleven were complete. The probable error of each set was calculated in the usual way, and the sum of these probable errors found to be $0\overset{\circ}{.}48$, or an average of about $0\overset{\circ}{.}035$ for a set. Taking $\theta=45°$, the percentage error of $\cos\theta$ is 0.05 per cent or 1 part in 2000.

Hence the experimental method should be capable of giving λ to at least 1 part in 1000, or to about 5 A.

Effect of the size of the apertures.—Hamy[1] derived a formula for the correction to be applied to observations made with apertures of finite width. The apertures were assumed to be rectangular, of width a and separation D: The source was assumed to be a uniformly luminous circular disk of angular diameter β. The fringes will disappear near the center of the pattern when

$$D=\frac{1.22\lambda}{\beta}\left\{1+0.765\left(\frac{a}{D}\right)^2\ldots\ldots\right\}.$$

---

[1] *Bulletin Astronomique*, **16**, 257, 1899.

No other cases were treated. It would seem reasonable to assume, however, that if the source is a double star of angular separation a, the corresponding formula would be

$$D=\frac{\lambda}{2a}\left\{1+0.765\left(\frac{a}{D}\right)^2 \ldots \ldots\right\}$$

since the correction must depend upon some property of the observing instrument. For circular apertures, a further modification would no doubt be required, and we may reasonably assume that this would be the substitution of $a'/1.22$ for a where a' is the diameter of the circular aperture.

Observations were made on the artificial double star with circular apertures of various sizes, using the method of rotating aperture plate. These gave the *same* value of θ for all sizes of apertures, well within the errors of observation, thus indicating that for this method of observing and with this kind of a source no correction whatever is required. This result was so surprising that it seemed worth while to make a theoretical examination of the problem, using a method somewhat different from that employed by Hamy.

The method adopted, while being very simple, has the advantage of giving a clear physical picture of the phenomenon so that the need for a correction, if required, is made evident.

Let us consider rectangular apertures having a width a, and a length b, and let their centers be separated by a distance D. When the line joining the apertures is parallel to the x-axis the intensity in the focal plane due to a distant point-source is

$$I=I_0\frac{\sin^2\frac{\pi ax}{F\lambda}}{\left(\frac{\pi ax}{F\lambda}\right)^2}\frac{\sin^2\frac{\pi by}{F\lambda}}{\left(\frac{\pi by}{F\lambda}\right)^2}\cos^2\frac{\pi Dx}{F\lambda} \tag{11}$$

If a double point-source is used, the angular separation being a, we have, writing $2c=Fa$

$$I=I'_0\left\{\frac{\sin^2\frac{\pi a(x-c)}{F\lambda}}{\left(\frac{\pi a(x-c}{F\lambda}\right)^2}\,\frac{\sin^2\frac{\pi by}{F\lambda}}{\left(\frac{\pi by}{F\lambda}\right)^2}\cos^2\frac{\pi D(x-c)}{F\lambda}\right.$$
$$\left.+\frac{\sin^2\frac{\pi a(x+c)}{F\lambda}}{\left(\frac{\pi a(x+c)}{F\lambda}\right)^2}\,\frac{\sin^2\frac{\pi by}{F\lambda}}{\left(\frac{\pi by}{F\lambda}\right)^2}\cos^2\frac{\pi D(x+c)}{F\lambda}\right\}\qquad(12)$$

If circular apertures of radius r be employed, we have equation (10) instead of the foregoing.

These expressions hold for monochromatic light of wave-length λ, and hence by definition (a) they are also valid for white light of visual effective wave-length λ. If we limit ourselves to the phenomena on the x-axis, which in no way restricts the generality of the treatment, we can place the factors involving y equal to 1 in equation (11) and write x for $\sqrt{x^2+y^2}$ in equation (10). Writing $D/a=n$, equation (12) may be put in the form

$$I=I'_0\left\{\frac{\sin^2\frac{z-c'}{n}}{\left(\frac{z-c'}{n}\right)^2}\cos^2(z-c')+\frac{\sin^2\frac{z+c'}{n}}{\left(\frac{z+c'}{n}\right)^2}\cos^2(z+c')\right\}\qquad(12a)$$

in which $c'=\pi Dc/F\lambda$.

When n is very large, the intensity near the center of the pattern is given very nearly, by

$$I=I'_0\{\cos^2(z-c')+\cos^2(z+c')\}.$$

The visibility will be zero when $z-c'=z+c'-\frac{\pi}{2}$, or, $2c'=\frac{\pi}{2}$, or, applied to equation (12), we find that for a small in comparison to D, the fringes will vanish when

$$\frac{2\pi Dc}{F\lambda}=\frac{\pi}{2}$$

whence $2c=F\lambda/2D$, or, since $2c/F=a$,

$$a=\frac{\lambda}{2D}$$

which is the ordinary formula, equation (2).

In order to form a clear idea of what happens when a is increased, let us refer to Figure 2, and for this purpose we will assume that the dotted curve represents

$$\frac{\sin^2\frac{z-c'}{n}}{\left(\frac{z-c'}{n}\right)^2}$$

instead of

$$\frac{4J_1^2\left(\frac{2\pi rx}{F\lambda}\right)}{\left(\frac{2\pi rx}{F\lambda}\right)^2}$$

which really makes very little difference.

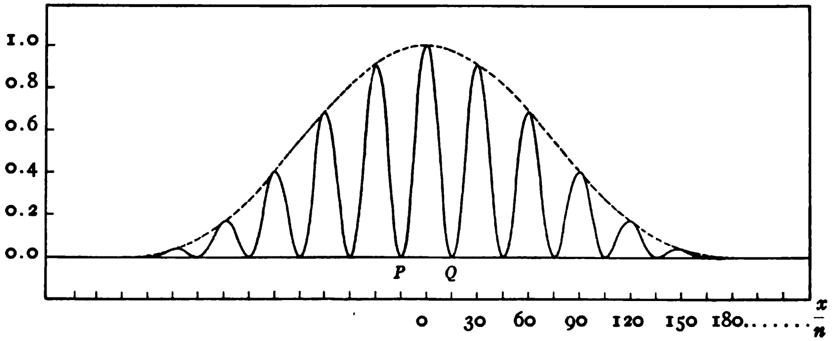

FIG. 2.—Dotted curve; $y=\frac{\sin^2\frac{x}{n}}{\left(\frac{x}{n}\right)^2}$ $(n=6)$

Full curve: $y=\frac{\sin^2\frac{x}{n}}{\left(\frac{x}{n}\right)^2}\cos^2 x$

With this understanding, we have then $2n$-loops of the inscribed curve in the central loop of the dotted curve. If n is very large, or a very small, the loops of the inscribed curve near the origin all have very nearly the same height, the dotted curve approaching a straight line parallel to the x-axis, and the inscribed curve approaching a true cosine-square curve more and more closely as

n is increased. If, under these conditions, we superpose two patterns displaced in the x direction by one-quarter of a period, the maxima of one will fall upon the minima of the other, and the sum will be a constant, which leads directly to equation (2).

When n is not large, the inscribed curve may be regarded as a distorted cosine-square curve, the distortion becoming more and more pronounced the more the dotted curve deviates from a straight line parallel to the axis. Since with white light the fringes are observable only near the center of the pattern, it follows that we need consider only the loops of the inscribed curve which are near the origin, and when n is small we have to confine our attention to the central fringe only, or to the portion of Figure 2 between the points P and Q. Let the ordinate at $x=0$ be taken as unity, and expressing x in degrees, we have for the undistorted curve ($n=\infty$) the value 0.500 at $\pm 45^\circ$ and the value 0 at 90°. For the distorted curve we have unity at $x=0$, and zero at 90°, but a value less than 0.500 at 45°. Hence if we superpose two such curves, 90° apart, there will be a minimum at 45°, or at the point midway between them, which will disappear only if their distance apart is made less than 90°. This makes it clear why equation (2) requires modification for small values of n, at least if we regard only the center of the pattern. It is necessary to remember, however, that when the minimum at 45° for the superposed curves disappears, the minimum at 135° has been made more prominent, and accordingly the "correction" that we employ to remove the minimum at 45° is really somewhat too large.

In order to calculate the angular displacement necessary to fill up the minimum at the center of the superposed curves, we need only equate the second derivative of (12) with respect to c, equate it to zero, and solve for c on the assumption that $x=0$. The factor in y is of course placed equal to unity. Table I gives the results of this calculation for a few values of n.

Here we have written the formula applicable to double stars in the form

$$D=\frac{\lambda}{2a}(1+K)$$

where K according to Hamy is $0.765\left(\frac{a}{D}\right)^2$.

Experimental verification of the results given in Table I.—In Table I we have applied Hamy's formula to a case which was not considered in his investigation, namely that of double stars. The difference between our calculated values and those given by Hamy's formula as here applied is so great that it seemed worth while to test both the case of double stars and that of a star disk experimentally, to see if there is really any difference between the two.

TABLE I

n	c	K(Calculated)	K by Hamy's Formula	Ratio $\frac{K\text{(Hamy)}}{K\text{(Calc.)}}$
∞	45°.000	0	0	....
11.5	44.9234	0.0017	0.00577	3.395
5.75	44.6938	0.0068	0.02314	3.403
4.60	44.5275	0.0106	0.03615	3.411
3.45	44.1721	0.0187	0.06427	3.437

The experimental arrangement was as follows: A screen having a horizontal slot 2 mm in width was placed in front of the objective of the observing telescope. A plate was arranged to slide in a vertical direction directly in front of and in contact with this screen. This plate had two parallel slots 5 mm apart, each tapering from ½ mm in width at one end to 2½ mm at the other end, the length being 55 mm. This arrangement provided two nearly rectangular openings in front of the objective, at a constant distance apart; by sliding the plate up and down the width of each opening could be varied continuously from ½ mm to 2½ mm, or, what amounts to the same thing, n could be varied from $n=2$ to $n=10$ in a continuous manner.

Observations were made both on an artificial double star, and on an artificial star disk. The disk was illuminated with approximately monochromatic green light, in order to avoid the variations in color which are troublesome with white light, especially when the visibility of the fringes is low.

The observations were made by finding the distance between the observing telescope and the source which gave the minimum visibility for $n=10$, and also the distance at which the central minimum for $n=2$ just vanished. Since the results were the same as far as the correction factor is concerned, whether a star disk or a double star was used, we will record only the observations made on the star disk:

Wave-length of green light (mean) 5400 A.
Diameter of artificial star disk (δ) 0.650 mm.
Distance between centers of rectangular apertures $(D)=5.0$ mm.
•Minimum visibility $n=10$, at 492 cm ± 2 cm.
Minimum in central fringe just disappears $(n=2)$ at 478 *cm* ± 2 cm.

Using $L=D\delta/1.22\lambda$ for the distance at which fringes should vanish with $n=\infty$ and substituting, we get $L=493.3$ cm, which, considering the uncertainty in the value of λ used, is sufficiently near 492 cm.

From this we find an experimental value of K for $n=2$, namely, $\frac{492}{478}-1$ or $\frac{493.3}{478}-1$, according to whether we use the observed or calculated value of L_∞. The first gives $K=0.0293$, the second $K=0.0320$, while by Hamy's formula, which is strictly applicable to this case, K should equal 0.19125. An extrapolation of the values in Table I to $n=2$ gives a value of K slightly larger than 0.03, which is in good agreement with the observations. We must conclude that the formula given by Hamy is incorrect, and that if the observations are made directly on the vanishing of the minimum in the central fringe, the values given in Table I are correct. It must be borne in mind, however, that if the observations are made simply on the minimum visibility of the pattern as a whole, which on account of seeing, will generally be the case in astronomical work, the simple formulae (1) and (2) are to be used without any corrections. It has already been shown (see p. 64) that, with the method of rotating apertures, no correction to the simple formula is required.

Determination of the effective wave-length of sunlight.—We will first consider some general questions which bear directly on the problem. It may be assumed that the quality of sunlight outside of our atmosphere is constant or very nearly so. In passing through the atmosphere it is modified by selective absorption and scattering to an extent which depends both on the thickness of the air layer and upon its character, such as water-vapor content, smoke, haze, etc. Most of these factors will reduce the intensity of the short wave-lengths more than that of the red end of the spectrum. Water-vapor is, however, an exception. It is very transparent to the short wave-lengths, while absorption begins somewhat below 6000 A and increases rapidly toward the red. Since its absorption lies almost entirely on the red side of the maximum of the sensitivity curve of the human eye, it appears that the effect of water-vapor by itself is to move the value of the effective wave-length toward the violet. Dust, smoke, haze, and increased thickness of the atmosphere due to large zenith distances might be expected to shift the wave-length to the red. A priori, then, a given observer would not expect to find a constant value for the wave-length.

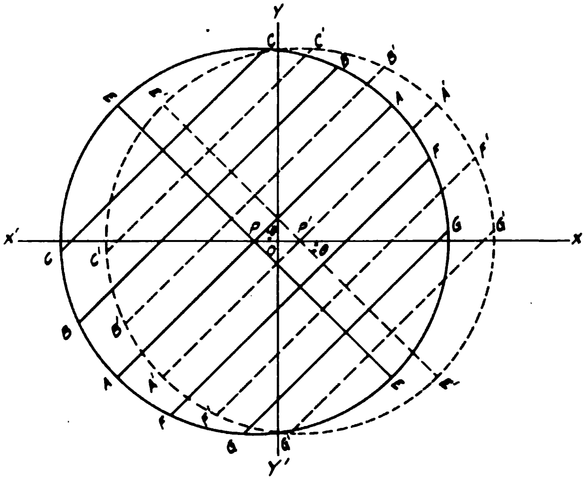

FIG. 3.—Pattern for double star

Apparatus.—Figure 1 gives a schematic diagram of the apparatus, and Figure 4 is a drawing of the observing telescope and aperture plate. The lens *C* is a theodolite objective of 22 mm aperture and of about 20 cm focus. The miscroscope consists of an 8-mm objective and a 7.5-power eyepiece, giving a linear magnification of about 125×. The circle *B* is graduated in degrees and can be read easily to 0°1. It carries the aperture plate *AA*, and is rotated by a worm *DW*, so arranged that one revolution of the worm gives a rotation of 10°. The whole instrument is mounted on a solid brass base plate, *K*, which is easily clamped to a table or other convenient support. The slot *S* was designed to hold a plane parallel quartz plate, cut at 45° to the optic axis. Viewed through this plate, a single distant pinhole appears as an artificial double star, whose angular separation depends only on the thickness of the quartz plate. A slight difficulty arises in the use of such a quartz plate, on account of the fact that one of the emergent beams of light is colored, due to the dispersion of the quartz. A great deal of time was spent in the calibration of the plate by the use of monochromatic light, and a few observations were made using it in conjunction with the 100-inch Hooker telescope on some of the brighter stars, but the results were not entirely satisfactory. When the quartz plate is not used the slot *S* is covered by a suitable cap. In the determination of the wave-length of sunlight the source was always a pair of pinholes made and mounted as described above; the sunlight was reflected into the quartz projection lens *L*, Figure 1, by a two-mirror coelostat, the mirrors being freshly silvered and polished every two or three weeks.

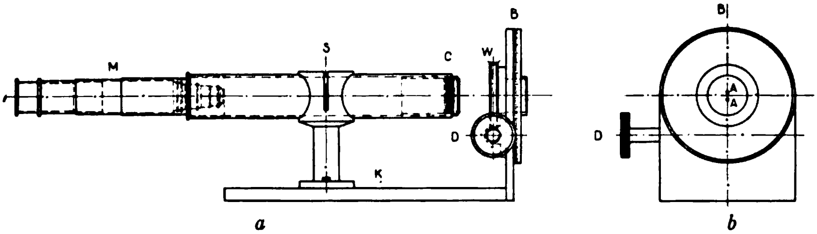

FIG. 4.—Laboratory apparatus
a. Side view. *b*. End view

In Tables II and III are collected the observations which have been made with the apparatus shown in Figure 4. Table II gives the observations made on Mount Wilson, June 15 to June 18, while in Table III are collected all the miscellaneous observations made under all sorts of conditions of sky in Pasadena. The first two columns give the date and hour of the observation. Since each complete observation required about 10 minutes, the time given refers to the middle of the observation. The third column gives the zenith distance of the sun to the nearest degree at the time. The fourth column gives the number of settings in each determination, which was always 20, excepting for the first observation on Mount Wilson which was stopped by fog, and for the three early observations on June 18, when the available time did not permit a complete set. The only observations calling for special comment are those of April 5 and 6 in Pasadena. These were made immediately after a rain, and the abnormally low value for the wave-length may be due to a relatively large amount of water-vapor; this, as mentioned above, would tend to displace the effective wave-length to the violet.

TABLE II

MOUNT WILSON OBSERVATIONS

Date	Mean Time	ζ	No. of Settings	λ	Remarks
June 15.....	8:35 A.M.	45°	16	(5546)	Incomplete. Fog blowing over. Image drifting badly.
June 16.....	5:14 A.M.	85	20	5631	
June 16.....	5:46 A.M.	79	20	5574	
June 16.....	6:35 A.M.	69	20	5542	
June 16.....	7:53 A.M.	53	20	(5535)	Fog coming over. Foggy remainder of day.
June 17.....	6:32 A.M.	69	20	5554	
June 17.....	7:52 A.M.	53	20	5515	
June 17.....	8:34 A.M.	45	20	5519	Faint haze.
June 17.....	9:18 A.M.	35	20	5519	
June 17.....	10:06 A.M.	25	20	5520	
June 17.....	10:49 A.M.	17	20	5515	Clouded over at 11:00 A.M.
June 17.....	4:50 P.M.	65	20	5546	Clouds. Haze as high as mountain tops. Sky blue.
June 17.....	5:22 P.M.	71	20	5579	Clouds. Haze as high as mountain tops. Sky blue.
June 17.....	6:13 P.M.	82	20	5596	Through very thin clouds. Haze.
June 17.....	6:37 P.M.	86	20	5625	Haze only.
June 18.....	4:58 A.M.	87	8	5656	Sky clear all day.
June 18.....	5:06 A.M.	86	8	5629	Sky clear all day.
June 18.....	5:07 A.M.	85	8	5619	Sky clear all day.
June 18.....	5:17 A.M.	83	20	5593	
June 18.....	5:47 A.M.	78	20	5561	
June 18.....	6:12 A.M.	73	20	5546	
June 18.....	6:44 A.M.	67	20	5528	
June 18.....	7:43 A.M.	55	20	5523	
June 18.....	8:12 A.M.	48	20	5512	
June 18.....	9:10 A.M.	37	20	5512	
June 18.....	10:11 A.M.	25	20	5515	
June 18.....	10:56 A.M.	17	20	5513	
June 18.....	11:20 A.M.	14	20	5510	
June 18.....	11:58 A.M.	11	20	5509	

TABLE III

Pasadena Observations

Date	Mean Time	ζ	No. of Settings	λ	Remarks
March 27...	10:11 A.M.	40	20	5537	
March 27...	11:41 A.M.	32	20	5521	
March 28...	9:05 A.M.	48	20	5568	
March 28...	9:57 A.M.	42	20	5531	
March 28...	11:03 A.M.	34	20	5515	
March 28...	2:03 P.M.	42	20	5555	
March 29...	9:19 A.M.	48	20	5556	
March 29...	9:50 A.M.	42	20	5542	
March 29...	10:29 A.M.	36	20	5522	
March 29...	11:30 A.M.	31	20	5525	
March 29...	1:10 A.M.	36	20	5559	Sky very thick.
March 30...	8:30 A.M.	56	20	5562	Sky thick toward east.
March 30...	9:10 A.M.	49	20	5537	
March 30...	9:40 A.M.	45	20	5542	
March 30...	10:03 A.M.	41	20	5547	
March 30...	10:27 A.M.	37	20	5528	Clearing rapidly.
March 30...	11:01 A.M.	33	20	5522	
March 30...	11:25 A.M.	31	20	5515	
March 30...	1:00 P.M.	34	20	5519	
March 30...	1:30 P.M.	38	20	5522	
March 30...	2:05 P.M.	43	20	5526	
March 30...	2:50 P.M.	51	20	5526	Cirrus developing rapidly.
April 2...	10:10 A.M.	28	20	5553	Some fog. Light unsteady.
April 2...	10:38 A.M.	34	20	5547	White sky, haze.
April 2...	11:10 A.M.	31	20	5524	
April 2...	11:30 A.M.	30	20	5524	
April 2...	1:35 P.M.	37	20	5527	
April 4...	8:30 P.M.	55	20	5524	After a rain. Cumulus clouds. Sky clear. Windy.
April 4...	8:56 P.M.	50	20	5527	
April 4...	9:48 P.M.	40	20	5524	
April 4...	10:42 P.M.	34	20	5533	
April 4...	11:17 P.M.	31	20	5531	Observer P.W.M.
April 5...	9:10 P.M.	46	20	5517	Thin cirrus here and there.
April 5...	10:12 P.M.	37	20	5494	Through cirrus.
April 5...	11:13 P.M.	30	20	5503	Clear.
April 6...	8:05 A.M.	59	20	5515	Clear.
April 6...	9:46 A.M.	39	20	5513	Clear.
April 6...	11:22 A.M.	29	20	5506	Clear.
June 21...	8:50 A.M.	40	20	5560	Haze obscuring mountains. White area around sun. Sky blue otherwise.
June 21...	9:26 A.M.	33	20	5545	
June 21...	10:12 A.M.	24	20	5533	
June 21...	11:05 A.M.	14	20	5524	
June 21...	11:40 A.M.	11	20	5522	

Figure 5 gives a graphical representation of the Mount Wilson results. This shows that on clear days for purposes of interferometer measurement the wave-length is sensibly constant if the zenith distance is less than 60°, while for greater zenith distances it rises rapidly. The Pasadena observations in general show that for zenith distances less than 40° there is not much change in the wave-length. For small zenith distances the Pasadena observations indicate a value not far from 5520, while the Mount Wilson observations indicate 5510. The difference of 10 A, though perhaps real, is so small that it may be said to be negligible as far as ordinary astronomical interferometer work is concerned.

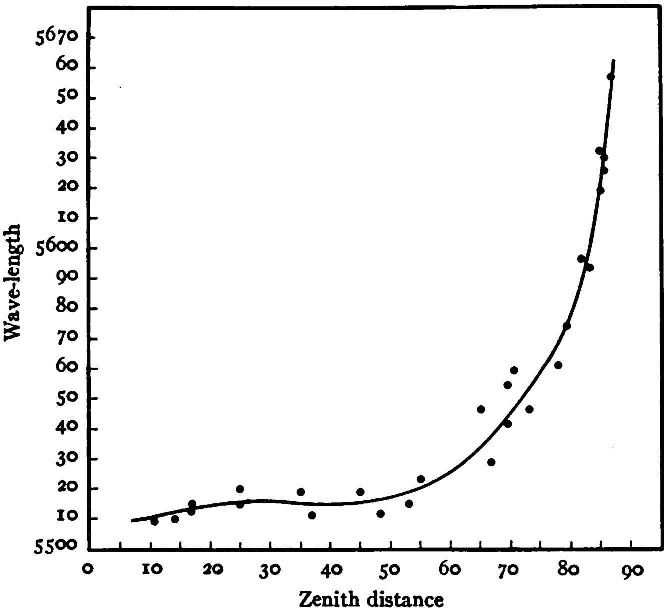

Fig. 5.—Mount Wilson observations
June 16–18, 1921

Mount Wilson Observatory
October 1921

Minor Contributions and Notes

ON THE SYSTEM OF PROCYON

ABSTRACT

Radial and orbital velocities of the system of Procyon.—A series of forty-five plates made from 1908 to 1913 give for V_0, after correction for the orbital velocity, the value -3.52 ± 0.057 km/sec. Taking the parallax at $0''.33$ and the masses as 2.70 and 0.89, K comes out 0.60 km/sec. The sign of the inclination is assumed to be positive, but is not definitely known. In order that more accurate information as to the orbit may be obtained, it is hoped that other observatories will co-operate in measuring radial velocities during the next few years, since as favorable an opportunity will not occur again for thirty-nine years.

The most recent orbit of this well-known and interesting binary is due to Lewis Boss[1] who gave the following elements of the orbit of the faint star about the brighter one:

$$
\begin{aligned}
T &= 1886.5\\
P &= 39^{\mathrm{y}}.0\\
\Omega &= 150^{\circ}.7\\
i &= \pm 14^{\circ}.2\\
\omega &= 36^{\circ}.8\\
a &= 4''.05\\
e &= 0.324
\end{aligned}
$$

Aitken[2] gives the masses of the two stars in terms of the sun's mass as $m = 2.70$, $m' = 0.89$.

The following values of the parallax have been published: L. Boss[1] (adopted) $0''.33$; Kapteyn and Weersma[3] $0''.324$; Adams,[4] spectroscopic, $0''.347$; trigonometric, $0''.309$. There is considerable uncertainty in the foregoing elements owing to the difficulty of the observations and the short length of the arc described by the faint companion since its discovery by Schaeberle in 1896.

As observations of radial velocity, if sufficiently numerous and differential, promise results of value, it is opportune at the present time to call attention to them and to invite the co-operation of other observatories in an attack on the problem of the spectroscopic orbit.

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[1] *Preliminary General Catalogue of 6188 Stars.*

[2] *Popular Astronomy*, 18, 485, 1910.

[3] *Groningen Publications*, 24.

[4] *Astrophysical Journal*, 53, 51, 1921.

Cape observations point to a + sign for the inclination; a series of plates taken during the next apparition, when the ascending node is being closely approached, will remove any doubts.

Assuming the sign positive we have the following values for V_0 based on unpublished details of my measures made from 1908 to 1913, viz.:

Mean Date	Number of Plates	Radial Velocity	Correction for Orbital Velocity	V_0
		km	km	km
1909.35...	13	−3.74	+0.19	−3.55
1910.73...	7	−3.74	+0.13	−3.61
1911.46...	12	−3.51	+0.09	−3.42
1912.49...	13	−3.56	+0.04	−3.52
Means 1911.03...	(45)	−3.63	+0.11	−3.52±0.057

Adopting −3.5 km/sec. as the value of V_0, $0''.33$ as the parallax, $\frac{m}{m+m'}$ as $\frac{0.89}{3.59}$, and the elements of Boss, the value of K becomes 0.60 km/sec. and the following radial velocities are calculated:

Date	$i+$	$i-$	Difference
	km	km	km
1921.0..........	−2.87	−4.13	+1.26
1922.0..........	−2.79	−4.21	+1.42
1923.0..........	−2.75	−4.25	+1.50
1923.5..........	−2.74	−4.26	+1.52 (max.)

It is important to take the opportunity of observing the radial velocities at the approaching node, an opportunity which will not again occur for thirty-nine years, in order to obtain a more accurate value of K when the observations are combined with others made at the node due in 1938. During the next seventeen years the velocity-curve shows the most rapid variations in the radial velocity. It is proposed to take three plates on each of seven nights for the determination of a normal place. Corrected for the solar motion, the system is approaching the sun with a velocity of 19 km/sec. in a path inclined 14° to the line joining sun and star.

JOSEPH LUNT

ROYAL OBSERVATORY
Cape of Good Hope
September 3, 1921

PLATE I

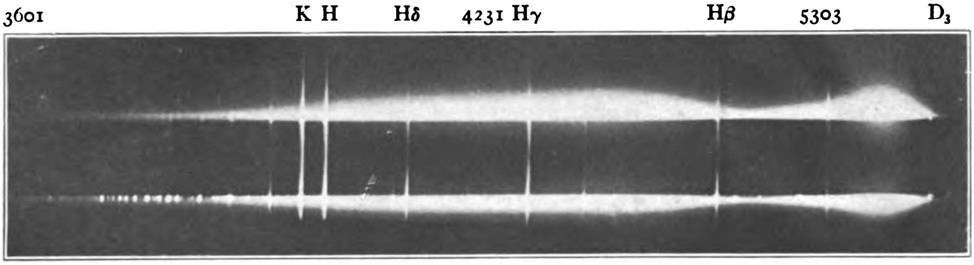

Fig. 1.—General spectrum of the corona. Single prism. Top W

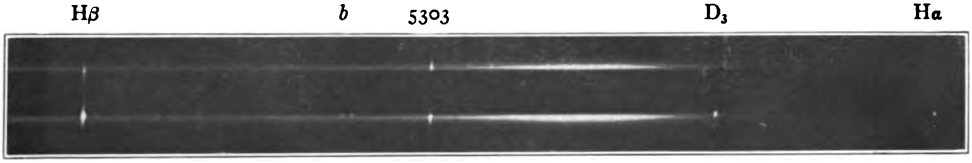

Fig. 2.—Spectrum of inner corona. Three prisms. Top E

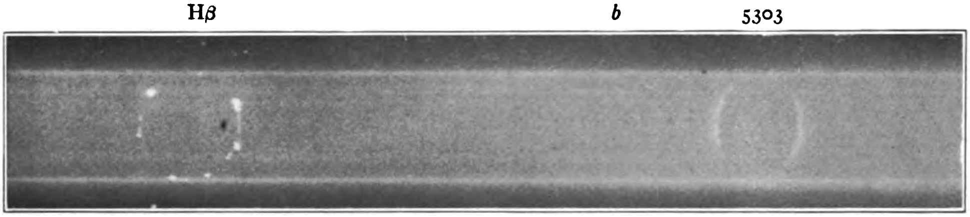

Fig. 3.—Hβ ring and green coronium ring. Top N

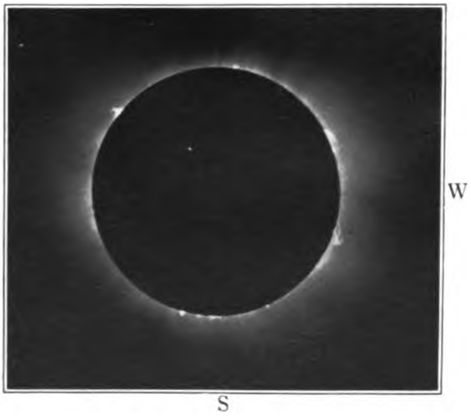

Fig. 4.—Prominences and inner corona with 39-foot camera.

Eclipse of June 8, 1918

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THE SPECTRUM OF THE CORONA AS OBSERVED BY THE EXPEDITION FROM THE LOWELL OBSERVATORY AT THE TOTAL ECLIPSE OF JUNE 8, 1918

By V. M. SLIPHER

ABSTRACT

Spectrum of corona observed at total eclipse of June 8, 1918.—The Lowell Observatory party, located near Syracuse, Kansas, obtained photographs with the following four spectrographs: (1) with single prism, 49 cm camera, slit E–W; (2) with single prism, 8.3 cm camera, slit N–S, then E–W; (3) with three prisms, 9-inch camera, slit E–W; (4) with three prisms, edges N–S, 47 cm camera, no slit. Weather conditions somewhat interfered with preparations; and thin haze veiled the sun during the eclipse, causing some scattering of light that rendered less certain the interpretation of some of the results. Yet useful plates were obtained with all the instruments. These show the usual prominence and chromospheric lines, the chief lines of coronium, the continuous spectrum of the corona, with weak dark lines whose origin is not quite certain; and strips of continuous spectrum coincident with certain prominences and sections of the inner corona uncovered by depressions of the moon's limb.

Distribution of coronium.—The coronium line λ5303.0 indicates that this substance extended about a solar diameter above the sun's surface. The slitless spectra show many condensations in the green coronium ring. These are not related to the prominences. Coronium was fairly abundant generally along the sun's limb under those major extensions of the outer corona, and scarce where the coronal rays typical of the polar regions occupied the limb.

The Lowell Observatory station for observing the solar eclipse of June 8, 1918, was located about eight miles north and one mile east of Syracuse, Kansas. Points along the path to the west of Denver offered somewhat better conditions as regards weather prospects, higher altitude of the sun and somewhat longer period of totality; but the advantages of accessibility and wider distribution of the expeditions along the eclipse path were sufficiently weighty to determine upon a station where the main line of the

Santa Fe Railroad was intersected by the eclipse path in western Kansas. This region is only one day by direct train from Flagstaff. Thus shipments of apparatus could be depended upon to go through more promptly and with less danger of injury than where changes in transit are involved. Then, too, this choice placed the Lowell station the farthest eastward of any of those prepared to undertake general observations of the eclipse.

In the autumn of 1917, Messrs. C. O. Lampland and E. C. Slipher looked over the general conditions in western Kansas with a view to finding a station site, and early in May, 1918, Mr. E. C. Slipher went to Syracuse, chose a site, found its latitude and the meridian and made all necessary arrangements, secured the needed help and materials, and went forward with the preparations. Others proceeded to Syracuse later as their rôles required, until the membership of the expedition was constituted thus: Mr. and Mrs. V. M. Slipher, Mr. and Mrs. C. O. Lampland, Mr. E. C. Slipher, Mr. Stanley Sykes, and Mr. E. C. Mills, of the Lowell Observatory; Mr. George R. Agassiz, of Boston, Massachusetts, and Mr. and Mrs. O. H. Truman, of the University of Iowa; and, on June 8, Mrs. Percival Lowell joined the expedition for the eclipse.

During the last week preceding eclipse day the weather was, for the most part, unfavorable. There were frequent showers and the sky was clouded much of the time, day and night. From what had been learned from weather records it had been expected that some clouds and rain would be encountered, but not so much as was experienced the last week. Thus the preparations and rehearsals could not be completed in the manner originally planned.

On eclipse day the sky at sunrise was only partly clear, and soon became completely overcast. However, in the late forenoon the clouds dissipated and by mid-day the sky was mostly clear. Throughout the afternoon some veil-like clouds of haze were present and at eclipse time one of these covered the region of the sun. This haze somewhat reduced the light and rather seriously limited the effectiveness of the small-scale cameras in recording the extreme outer extensions of the corona, by its direct action, and also by its scattering the light of the prominences and bright inner corona over the faint outer parts of the corona. However, although the haze must have affected to some extent the photographs of the inner corona and the spectrograms, its ill effects were not always manifest. Indeed, the two large-scale cameras, particularly, secured a number of negatives of the brighter parts of the corona that in comparison with other eclipse results are seen to be of a high degree of excellence. This valuable and extensive material will be published later by Messrs. Lampland and E. C. Slipher, who were responsible for that part of the eclipse program and deserve full credit for the success of the results.

It was thought wise to confine the studies of the spectrum to that of the corona. For this work serviceable instruments could be arranged from the stellar and nebular spectrographic equipment of the observatory. High linear dispersion was not considered vital in the observations to be undertaken and in arranging the different spectrographs questions of choice between dispersion and light-power were decided to favor light-power, in order that the instruments might be quick enough under good conditions to get a strong record of the spectrum and still to get some record even should the conditions be poor. The mountings of the instruments were made largely of wood, but metal was employed where most needed to give the optical parts proper support. The weather at Syracuse made clear the weakness of wood and the wisdom of making all essential parts of metal. At the eclipse camp Mr. O. H. Truman assisted me for several days before the eclipse, and he and Mrs. Slipher shared with me the operation of the instruments during the eclipse.

The single-prism spectrograph.—A spectrograph of one 63° flint glass prism and a 380-mm camera of 33-mm aperture was used in recording the general spectrum of the inner corona from D to the ultra-violet. It was supplied with an image-forming lens of about the same aperture and focus as the collimator, which was of 490-mm focus and 32-mm aperture. The slit was east and west over the sun's diameter and long enough to include the corona to some distance off the east and west limbs of the sun. The exposure ran from count 2 to 78 of the 83 seconds of the total phase. A Cramer Isochromatic plate was used, and recorded the spectrum from D_3 to about λ 3650 in the ultra-violet.

The exposure began soon enough after second contact to register a great number of chromospheric bright lines on the east edge of the sun, but it stopped too early to show these on the west edge.

The prominence lines of calcium, hydrogen, and helium are long, especially so those of calcium. The chief coronium line, that near λ 5303.0 is also strong and, like the lines mentioned above, it extends across the space occupied by the moon's image, owing to scattering of the light in the earth's atmosphere. The coronium line near λ 4231 is present and also weak impressions of the lines λ 4086 and 3601. The continuous spectrum of the inner corona extends from D almost to λ 3650, with the general variations of intensity that are to be expected from the variations in color sensitiveness of the photographic plate used. In its strongest portion—in the blue—it is recorded outward from the sun's east limb to a distance of about 10 minutes of arc and to about 14 minutes from the west limb. In the outer margin of the continuous spectrum there are irregularities, especially between Hδ and λ 4500, which I am more inclined to believe are the weak beginnings of the dark-line solar spectrum rather than that they are bright lines of a true coronal origin. This plate is reproduced in Figure 1, Plate I.

Low-dispersion spectrograph.—A slit spectrograph of high light-power having a 66° prism of dense but clear flint glass and a camera of 35-mm aperture and 83-mm focus, was used in photographing the spectrum of the corona to as great distances from the sun's surface as possible. This spectrograph with its image-forming lens was mounted so that it could be rotated through 90° in order to obtain two spectrograms: one with the slit N–S, for the outer corona off the north and south limbs of the sun; the other, with slit E–W, to cover the corona to the east and west of the sun. Thus, with a long slit it was planned to compare the spectra of two of the most dissimilar parts of the corona, the polar, and the equatorial regions. The slit was long enough to cover the corona out on each side to about 1.2 times the solar diameter. Unfortunately we did not have time, under the weather conditions, to perfect the adjustments for rotation so as to have the slit cover the desired portions of the corona for the polar and the equatorial comparisons.

In consequence, for the first exposure the slit which was east and west crossed the moon's image near the south limb. The exposure started two seconds after the second contact and continued during the first half of totality. Then at mid-totality the instrument was rotated through 90°—which brought the slit north and south and about 10′ west of the sun's west limb—the plate was shifted and the second exposure continued up to within about two seconds of third contact.

The plate used was a Cramer Trichromatic. It has a good impression of the continuous spectrum, over the full length of the slit, and from the violet down to the D line. The exposure with the slit crossing the sun's image near the south limb shows many prominence and chromospheric emission lines, including the red line of hydrogen. The green coronium line is clearly recorded by the first exposure and faintly by the second. This record of the coronium line will be of interest later in connection with the slitless spectrogram. A few of the strongest groups of solar absorption lines are discernible toward the edges of the spectrum of the first exposure and traceable across the full width of the second one.

The variation of intensity of these spectra across their width is in full accord with what the direct photographs indicate for light intensities along lines over the corona occupied by the slit: i.e., the first exposure slit E–W, shows more extension and density to the west than to the east of the sun, and the second more uniform density across its width. The ends of the slit for the first exposure reached distances of about 1.2 diameters from the sun's limb and for the second exposure about 1.3 diameters. The two exposures overlap by about one-eighth of their combined width and the plate was somewhat fogged by extraneous light from too early drawing of the slide. These faults and the incorrect placing of the slit for the two spectra made them unsuitable for the comparison of the coronal spectrum of the polar with the equatorial region.

The presence of the cloud of haze over the sun raises a question as to the extent reflected sunlight contributed to the dark-line spectrum. The direct photographs and also the slit spectra bear evidence of light scattered from the prominences and bright inner corona. The great extension of the calcium and hydrogen prominence lines and their presence over the moon's image is clear proof that prominence light was widely scattered. Some years ago the late Lord Rayleigh showed how rapidly the power of our atmosphere in scattering sunlight increased as the wave-length was decreased, and previous eclipse spectra have shown the effect with clear skies. Hence our air must have greatly aided the cloud in scattering the violet rays among which the dark lines, in the present case, were observed. It does not seem possible to determine just how great the effect of the haze was, but the presence of the prominence emissions over these spectrograms leaves no doubt that the strictly continuous spectrum of the bright inner corona was likewise spread over that of the outer corona, which would tend to cover up any dark lines it possessed. It is not possible to know to what extent these two opposing effects of the haze balanced each other, and hence we cannot say certainly that the dark lines observed were of true coronal origin although they appear to be such.

Three-prism spectrograph.—The Brashear stellar spectrograph of three prisms, supplied with a camera of nine inches focus, was adjusted for the visual end of the coronal spectrum with the hope of photographing more of the red region than is usually done. I had planned to sensitize a plate to the extreme red, but conditions did not permit my doing that and a Cramer Trichromatic plate was finally used.

The exposure ran from count 2 to 80. The slit was placed east and west. The plate records well the continuous spectrum from the blue down to D_3, where it rapidly weakens owing to the diminished sensitiveness of this emulsion in that region. On the shorter wave-length side of D_3 there are present the usual hydrogen and helium prominence lines and also a number of chromospheric lines. The C line of hydrogen is registered, but no other bright lines are evident on the red side of the D_3 line. The green coronium line λ 5303.0 is strongly impressed upon the plate (Fig. 2), but I have recognized no other true coronal lines.

The green line is long and nearly equally intense on the east and the west of the sun. Prominence and chromospheric lines, however, are much stronger on the east than on the west side, implying that the exposure began a little nearer second contact than the ending was to third contact. The comparison spectrum was unfortunately accidentally omitted from this plate, but the wavelength of the green line was determined with fair accuracy from measures of it relative to nearby prominence and chromospheric lines, the resulting value being λ 5303.0. Owing to scattering of light the green coronium line, like $H\beta$ and $H\gamma$, is traceable across the moon's image.

The slitless spectrograph.—A battery of three 63° dense flint prisms and a camera of 35-mm aperture and 472-mm focus served as a prismatic camera for investigating the spectrum of the corona with special regard to the green coronium ring. The adjustments were such as to include the spectrum from D_3 to beyond F in the blue, in order to have these helium and hydrogen radiations for comparison with the coronium ring in studying the distribution of this substance around the sun's surface. The prisms were not at minimum for λ 5303 and the green ring is somewhat elliptical. The edges of the prisms were set north and south instead of east and west, the more usual position. This orientation was the more favorable for recording the coronal ring clearly along the east and west limbs where it shows most intensity and detail. In this case the east and west limbs are in the middle of the spectral band and the north and south limbs at its edges, where details are of course confused. On account of the persistent clouds on the last few days and nights I did not get the focus sharply adjusted; and, too, the instrument received a slight jar when the small spectrograph described above was rotated at mid-totality, the two being mounted on the same polar axis. In consequence the objective spectra are not quite sharp, but it seems that nothing of importance was thereby lost.

The solar image on the plate of the prismatic camera had a diameter of 4.3 mm.

Three exposures were made with this instrument on isochromatic plates; one from the signal "go," for three seconds; the second (on the same plate shifted) for seventy seconds; and the third (on a separate plate) for about a second near third contact, for the flash. This last exposure was a little late and owing to the low dispersion is overexposed and probably does not merit any further consideration and description here.

The first of these exposures records the eastern crescents of F and D_3 together with a faint impression of the green coronal crescent. The second, the long exposure, secured a good impression of the green ring and of the irregular, broken rings of hydrogen F and helium D_3. The plate is of course considerably fogged by scattered light and by the light of the bright inner corona. There is sharp increase in the intensity of the continuous spectrum of the corona at its edges corresponding to the north and south limbs of the sun. These margins are strikingly sharp. They of course mask the coronium ring near the north and south points of the sun's disk; but as the ring becomes weak before reaching the edges except in the southwest quadrant, probably not much detail of the ring was thereby covered. However, it is strongly advised in such observations of future eclipses that the observer provide himself with a pair of instruments of this simple type and mount them so that for one the refracting edge of the prism (or grating rulings) is north and south, and for the other east and west. In this way, between the two, he will secure the whole of the sun's limb without masking, and for the most part he will thus also secure the much-to-be-desired duplicate records of the coronium spectrum.

The second exposure, with this instrument, of seventy seconds, began and ended about equally near second and third contacts, respectively, but it ended a very little nearer third contact. This negative shows that the green coronium ring is not uniformly intense around the sun. Its strength is much greater along the east and west limbs, generally, than about the north and south ones, yet its stronger parts do not coincide with the equatorial regions or its fainter parts with the polar regions. It occupies a greater extent of the west limb than it does of the east one, and on that side it extends in considerable strength farther to the south of the equator than it does to the north, in fact it seems traceable to the south pole. On the east limb, on the other hand, the ring stops quite suddenly nearer the equator on the south than on the north. That is, the coronium ring is faint or absent along the sections of the sun's limb occupied by the bristling streamers typical of the polar regions. The spectrogram, compared with direct photographs, shows that the coronium substance was relatively abundant generally in those regions of the sun's limb from which flowed the great extensions of the corona. While there seems to be this general correspondence in the distribution of the coronium substance and the main features of the corona, the nature of the relationship is a problem for further investigation.

In addition to these main variations in intensity of the coronium ring it has numerous local irregularities, its stronger portions being found in the zones of spots and prominences, generally; yet these do not coincide with individual prominences. The presence on the same plate of the F ring of hydrogen and the D_3 ring of helium, one on either side of the coronium ring, rendered comparison and study easy and definite. It is evident at a glance that the coronium ring is not at all like those of hydrogen and helium. The distribution of coronium thus is different from that of hydrogen and helium, and so no obvious relationship of coronium to these other substances is indicated. It is true only in a general way that the green ring is strong throughout regions including the zones frequented by sun-spots and prominences. The condensations in the coronium ring, none of which are definitely limited, stand in striking contrast to those of hydrogen and helium due to the prominences. The spectrogram shows the green radiation to fade, at first rather rapidly, outward from the sun's surface. While this negative gives no direct evidence on the question whether or not coronium is concerned in the arches over the prominences, it reveals a distribution of coronium favorable to its presence there. The coronium line on the plate of low dispersion, already described, extends outward fully a solar diameter into the corona. It is stronger off the southwest limb than off the southeast one—both of which were similarly included by the slit—thus conforming to the intensity-distribution of the coronium ring shown on the objective-prism plate. Of course we must consider here the effect that scattering of light in the earth's atmosphere has upon the extension of the line. Some assistance on this point is furnished by the negative for the general coronal spectrum, the first to be described of the series. This negative contains a good record of the emissions of hydrogen, helium, calcium, and coronium from the yellow to the ultra-violet, and, as it is without a comparison spectrum, it is well adapted to comparing the emission lines inside the moon's limb, where scattering is wholly responsible for their presence, with their extensions outward from the same limb into the corona, where the effect is a combination of scattering of light and the distribution of the substances involved. Inside the moon's disk the coronium line is as weak as, or weaker than, the helium line λ 4472 and, of course, weaker than any of the hydrogen lines; but when compared with the same lines at, say, half the solar radius outside the moon's limb, on the west, the coronium line is as strong as the strongest hydrogen line. Lord Rayleigh investigated the scattering of light in the atmosphere and found it to increase inversely as the fourth power of the wave-length. The effect is clearly disclosed by this spectrogram, the coronium line showing less spreading over the moon's image than do the blue and violet lines of hydrogen and helium. But outside the sun's limb over the corona something more than scattering is at work, for the coronium line gains strength relatively to these same H and He lines of shorter wave-length. This seems to imply that some coronium was present high above the sun's surface, as high apparently as the arches over the prominences.

These arches seemed unusually well developed at this eclipse and it is unfortunate that the presence (or absence) in them of coronium could not be cleared up. The presence of the coronium line high above the sun raises the question whether its luminescence may not be, as has been suggested, an electro-magnetic phenomenon similar to that of the aurora: the arches over the prominences have some resemblance to magnetic lines of force. One might further remark upon the fact that coronium and "aurorium" each has its distinguishing radiation in the same general region of the spectrum. Much remains to be learned of coronium and its relationships, at future eclipses.

This objective-prism plate, Figure 3, has further interest in that it shows strips of well-defined, continuous spectrum which run longitudinally through the main spectral background of the inner corona. Three of these are quite distinct and rather definitely limited in width. It appears evident that two of them coincide with the two strong prominences of the northern hemisphere, one on the east limb, the other on the west. The third is in the southern hemisphere and is somewhat wider and apparently related to some small prominences and an elevated region of the photosphere. Comparison of the maxima of the coronium ring with those of the H and He rings with regard to the strips of intensified continuous spectrum led me to the conclusion that the bands are related to the prominences and photosphere and not in any direct way to the substance of the green ring. In addition to what has been stated, the facts pointing to this conclusion are that the maxima of the ring are not definitely limited, as are the strips of continuous spectrum and there is not that correspondence between the two that would be expected were the ring maxima and the strips due to the same origin. It is noted that where the bands cross the ring it is generally denser, but it seems not more so than would result from the action of the blended light of the two.

It seems to be definitely established that some prominences possess a continuous spectrum, and brief computation shows that a depression of the moon's limb, moderate in depth among those known to exist[1], would have been equivalent to starting the exposure as much as 5 seconds early—or to stopping it that much late—which is ample time, near second or third contact, to expose sections of the photosphere and so introduce continuous spectrum. Observations clearly bear this out, as such lunar depressions have caused sharp strips of continuous spectrum on eclipse spectra in numerous cases. Both the causes mentioned above seem to have been concerned in the strips of continuous spectrum described here for the 1918 eclipse.

To Mrs. Percival Lowell, for her generous provision for the expedition and interest in its success; to Mr. Guy Lowell, the Trustee of the Lowell Observatory, who early proposed the undertaking; to Professor W. W. Campbell, Director of the Lick Observatory, for expert advice on plans and equipment; to Professor

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[1] *Astronomische Abhandlungen der Hamburger Sternwarte in Bergedorf*, 3, No. 1; *Selenographische Koordinaten von Friedrich Hayn.* IV Abhandlung des XXXIII Bandes, K. Sternwarte, Leipzig; *Astronomische Nachrichten*, No. 4615 (193, 118, 1912).

John A. Miller, Director of the Sproul Observatory, and Professor W. A. Cogshall, Director of the Kirkwood Observatory, for counsel and the loan of apparatus; to Mr. George R. Agassiz, Boston, Massachusetts, for the loan of a large lens; to the Naval, Flower, and Haverford observatories, for the loan of equipment, and to a number of citizens of Syracuse for their kind interest and assistance before and on eclipse day, it is a pleasure to make grateful acknowledgments.

LOWELL OBSERVATORY
FLAGSTAFF, ARIZONA
October 1921

SPECTROPHOTOMETRIC METHODS FOR DETERMINING STELLAR LUMINOSITY[1]

By BERTIL LINDBLAD

ABSTRACT

Two spectrophotometric methods for determining absolute magnitudes.—(1) *Relative intensity of spectral regions on either side of* λ *3907*. Using the 10-inch Cooke refractor with 6° objective prism, a series of images of decreasing density was obtained for each star on each plate, making each exposure 0.707 times the preceding, and from these images the exposure ratio ($E=t_2/t_1$) required to give the same density in the two adjacent spectral regions was determined for stars in well-known groups with common proper motions and hence common parallaxes—the Hyades, the Pleiades, the Ursa Major stream, and Praesepe—and also for some stars with known spectroscopic parallaxes, in all, 107 stars of types A and B and 74 stars of later types. The results show that, especially *for stars of type B8 to A3*, there is a definite correlation between E and M, the variation being such that even with the small dispersion here used the probable error of a determination of M for a star within this range of type and of absolute magnitude 0 to 4 is only $0^{M}4$. (2) *Relative intensity of cyanogen bands and adjacent spectral regions.* Using the same instrument and method as above, the relative density on either side of λ *3889* was determined for 16 stars of known parallax, and (since the band at λ 4216 though weak is more favorable for the redder stars) the relative density in regions λλ 4144–4184 and λλ 4227–4272 was determined for 78 late stars. The absorption was found to be strongest *for the giants of types G5–K5* and much weaker for dwarfs; in fact all stars with log $E>0.17$ are giants. Figures 3–5 show the variation of log E with M for various types.

Relation of spectra of A and B stars to absolute magnitude.—A study of some spectra made with the 60-inch showed that the increase in the density of the region λλ 3889–3907 with reference to λλ 3907–3935 is caused by a widening of the $H\zeta$ line and also of some arc lines of Fe and Si, which may be due to a greater density gradient in the outer layers of the less luminous stars of this type. α Cygni and the faint companion of o_2 Eridani represent extreme types.

Spectral type and luminosity of 137 faint stars of magnitudes 10 to 14, in Messier 11, Messier 13, and Selected Areas 63 and 64, were determined from 10 plates made with a small focal-plane spectrograph attached to the 60-inch. Dwarfs and giants were distinguished by applying the spectrophotometric results reported above.

Spectrophotometric parallax determinations.—The concordance of the results for log E as a function of M, as determined from the various groups of stars, indicates that the assumed parallaxes for the *Hyades, Pleiades, and Praesepe*, $0\overset{''}{.}023$, $0\overset{''}{.}010$, and $0\overset{''}{.}0072$, respectively, are accurate to within 10 per cent. The value found for *Messier 11* is $0\overset{''}{.}00025$.

Schwarzschild's constant p for Seed 23 plates was determined for images made with the 10-inch refractor and 6° prism and found to be equal to: $0.97-0.026\ (m-5)$, where m is apparent magnitude ranging from 4 to 8. X-ray developer was used.

I. A CRITERION OF ABSOLUTE LUMINOSITY FOR STARS OF SPECTRAL TYPE A

The faint companion of the star o_2 Eridani of apparent magnitude 9.5 and absolute magnitude $+10.3$ was found by Adams[2] to be of spectral type A0, and, intrinsically, is by far the faintest A-type star known. Some photographs of this object, taken with an objective prism on the 10-inch refractor and with a focal-plane spectrograph in conjunction with the 60-inch reflector, showed some peculiarities in the energy-distribution of the spectrum, when compared, for instance, with the spectrum of Sirius. The most conspicuous feature is the very strong widening of the wings of the hydrogen lines, connected with a decided weakening of the high-numbered members of the Balmer series. On the spectrograms of low dispersion taken here, $H\eta$ is the last hydrogen line to be identified with certainty. Seemingly as a consequence of this weakening of the extreme hydrogen lines, the photographs show a faint extension of the continuous spectrum much farther into the ultra-violet than is the case for an ordinary A-type star. A further difference between this star and Sirius was found in the distribution of energy between $H\epsilon$ and $H\zeta$, apparently consisting in a considerable decrease of intensity in the region λ 3889–λ 3907 as compared with the region λ 3907–λ 3935. This phenomenon may be caused partly by the expansion of the wings of $H\zeta$, but there was seemingly also some other cause; in particular, a widening of the silicon arc line λ 3905.6 was suspected. This line is well developed in the spectrum of Sirius, though probably it is to some extent a blend with an enhanced line of chromium. In order to see whether the last mentioned phenomenon might be a general criterion of intrinsic brightness for A-type stars, an investigation was begun on the Hyades which was later extended to include other groups of common proper motion.

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[1] *Contributions from the Mount Wilson Observatory*, No. 228.

[2] *Publications of the Astronomical Society of the Pacific*, **26**, 198, 1914.

The instrument used was the 10-inch Cooke refractor with an objective prism of 6°. The scale of the spectra thus obtained, about 340 A per mm in the region of the K line, is probably near the lower limit possible for a successful investigation of the effect. No widening of the spectra was undertaken. Satisfactory images of an A star of photographic magnitude 6.5 were thus obtained in one minute. Seed 23 plates were used, with X-ray developer. Owing to the steep color-curve of the objective for the extreme violet, only a small region of the spectrum can be in good focus for a given setting; that used corresponded to the interval λ 3889–λ 3920.

In order to measure the decrease of intensity at λ 3907 the following process was used. For each object a series of exposures was made on the same plate with exposure-times decreasing in the ratio $\sqrt{2}:1$. Each star thus produced a sequence of images of decreasing photographic density; that is, a scale of images which may be represented by the numbers 0, 1, 2, 3, etc., corresponding to the exposure-times t_0, t_1, t_2, t_3 . . . . , where $t_i:t_{i+1}=\sqrt{2}$. The density in the region λ 3895 to λ 3907 of a certain image, for instance, the first, was compared with that in the region λ 3907 to λ 3925 of the following images. The point in the scale of images for which the density in the second region equaled that of the first region in the reference image, was estimated by interpolation to a tenth of an interval. Suppose the exposure-time corresponding to this "scale-reading" to be t. The logarithm of the ratio $t_0:t$, denoted by log E, is then a measure of the energy-fall in the spectrum near λ 3907. The successive images define a scale with intervals of logarithmic exposure-ratio equal to 0.15, and the "scale-reading" observed is therefore readily converted into log E. As many estimations as possible were made from each series of images, starting with each of the spectra in succession as a reference image. The process is analogous to that used by Seares[1] in determining the color of a star by the method of exposure-ratios. Distorted images, as well as very dense or very weak images, have not been measured. Whenever possible, measurements were made with the plate in two opposite positions. In some of the measurements, especially of very bright stars, like Sirius, for which sufficiently short exposures could not be made with the accuracy necessary to form a reliable scale, the value of log E was estimated directly from the difference of density between the two regions in the same image. Apart from the influence of possible changes of gradation from one plate to another, the two methods, if checked against each other, should give equivalent results. This follows from Kron's[2] results for the law of photographic density as a function of intensity and exposure-time. If the gradation in a time-scale is constant, a difference in density obtained during the same exposure for two different light-sources will always correspond to the same ratio of exposure-times giving the same density, independently of the order of magnitude of the intensities, in this case, of the apparent magnitude of the star.

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[1] *Proceedings of the National Academy of Sciences*, 2, 521, 1916.

[2] *Publikationen des Astrophysikalischen Observatoriums zu Potsdam*, 22 (No. 67), 1913.

The wave-length λ 3907, separating the two regions, is easily identified with sufficient accuracy from its position relative to the lines Hζ, Hε, and K. The abrupt fall of intensity near this wave-length for low-luminosity A-type stars was confirmed by these observations and also on spectrograms of higher dispersion, as will be seen later. With the low dispersion used, the rapid change in intensity leaves the density fairly uniform in the two regions, λ 3895–λ 3907 and λ 3907–λ 3925. In fact, the success of the process depends on this circumstance. On the other hand, the smallness of the dispersion entails relatively large accidental errors, thus necessitating the use of a rather large number of images in order to get a reliable mean result.

Because of the juxtaposition of the two regions in the spectrum, it is evident that the influence of slight changes in the sensibility-curve of the plate, as well as of a variation in zenith-distance within reasonable limits from one object to another, will be negligible. The same will hold for a difference of gradation for the two regions. Further, if we use the law of density in Schwarzschild's form, $S=\phi\ (It^p)$, we may assume the value of p to be the same for the two regions. Suppose I_1 and I_2 to be the acting intensities for the first and second regions, respectively, and suppose S_1, t_1, S_2, t_2, to be corresponding values of density and exposure-time for the two regions. Then for equal densities we have $S_1=S_2$, and

$$I_1 t_1^p = I_2 t_2^p$$

$$\log \frac{I_2}{I_1} = p \log \frac{t_1}{t_2} = p \log E \,.$$

If we use the same kind of plates and the same developer, we have reason to assume p to be tolerably constant, though experience shows that large deviations may occur.

Since we may certainly disregard any difference in gradation for the two regions, there cannot, according to Kron's results, be any change of log E with the density of the images, a result which seems to be confirmed by the measures. This follows from the parallelism of the curves log It against log I for different degrees of density in Kron's diagrams. Two values of log I giving the same density will demand the same difference of log t for all values of the density. There may be, however, a slight change of log E with the absolute value of the acting intensity, that is, with the apparent magnitude of the star, due to the bending of the curves mentioned, corresponding to a change of the exponent p with the intensity.

The method described here admits a determination of Schwarzschild's constant p for a certain intensity, as soon as we have on a single plate the series of images of two different stars whose magnitudes are accurately known. The method of measurement is simply to interpolate the density of the successive images of one star into the scale defined by the images of the second star. In this way p has been determined from the relation $p=\frac{\Delta \log I}{\Delta \log t}$ for nearly all of the plates used here; in some cases p was determined for very different intensities on the same plate. The results seem to confirm Kron's conclusions. There is a decided decrease of p with the apparent magnitude. The mean from five plates is

$$\frac{\Delta p}{\Delta m}=-0.026\pm 0.002 \text{ (P.E.)},$$

for photographic magnitude 6.5.

Assuming a linear relation between p and m to hold for the variation of intensity met with in this investigation, we have as a mean for the first seventeen plates measured $p=0.97-0.026\,(m-5)$. Individual plates were assumed to differ from this equation by a constant ϵ, which in so far as possible was determined for each plate. The probable deviation in p from the formula for one plate, computed as a probable error for the seventeen plates mentioned, was found to be ± 0.03. A few Seed 23 plates of a later emulsion give, in the mean, lower values of p. One plate of the Pleiades was discarded by reason of an abnormally low value of p. The values of log E were reduced, as far as possible, to $p=0.97$. As a rule the corrections were small.

From stars measured on more than two plates it appears that the probable error of log E from a series of four measured images on a single plate is ±0.025. The residuals of different plates indicate a probable systematic error of about ±0.02. A considerable part of the probable error in log E for one plate must therefore be due to causes common to all the stars on the plate taken in the same exposure. For the greater part these are probably connected with irregularities in the images caused by imperfect guiding. From the foregoing discussion it seems unlikely that varying quality of the plates contributes appreciably. In computing the final values of log E, weights were therefore assigned equal to the number of images measured on the respective plates.

Attention was given first to members of well-known groups of stars with common proper motions, the Hyades, the Pleiades, the Ursa Major stream, and Praesepe, for which values of the parallaxes have been derived from the stream-motion or other plausible considerations. The large number of early-type stars in these clusters makes it much easier to establish the relation of spectral intensity to absolute magnitude than is the case for later types. Tables I–IV contain the results for stars of these groups; in addition, Table V gives results for some stars of types B5 to B9 of Kapteyn's groups in Orion and Scorpius, and for the two stars α Cygni and Comp. o_2 Eridani, which may be considered as representing extreme limits for the luminosities of A-type stars.

For the Hyades the parallax used for computing the luminosities is that of Kapteyn-Küstner,[1] $\pi = 0\rlap{.}''023$. The Ursa Major stars are given according to Bottlinger.[2] Thirteen of the fifteen A-type stars which are certainly members of the group have been observed for this investigation. The absolute magnitudes of ϵ and ζ_1 Ursae Majoris, spectroscopic binaries with composite spectra of nearly equal intensities, have been increased by $+0\rlap{.}^{M}75$ in order to reduce to the absolute magnitude of each component. For the Pleiades a list of probable members brighter than $9\rlap{.}^{M}5$ photographic has kindly been communicated by Dr. Trumpler of the Lick Observatory. The value of the parallax adopted is Trumpler's[3] result, $\pi = 0''.010$, derived by means of the orbital motion of ten visual binaries belonging to the group. Some stars of the list are not measurable on the plates because of overlapping images. The maximum exposures for the four plates used were 30 minutes, 10 minutes, 2 minutes, and 30 seconds, respectively. For the Praesepe cluster Kohlschütter's[1] value $\pi = 0''.0072$ was adopted. This was derived by a comparison of the relations between apparent magnitude and spectrum for the early-type stars in the Hyades and Praesepe. This value is at least consistent since it gives for the four bright K stars of Praesepe a mean absolute magnitude nearly equal to that of the corresponding stars in the Hyades. For one of the Praesepe stars mentioned, B.D.$+20°2166$, Mr. Adams has kindly secured a spectrogram, from which he estimates the absolute

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[1] *Publications of the Astronomical Laboratory at Groningen*, No. 23, 1909.

[2] *Astronomische Nachrichten*, **198**, 153, 1914.

[3] *Publications of the Astronomical Society of the Pacific*, **33**, 214, 1921.

TABLE I

HYADES, $\pi = 0''.023$

STAR		VIS. MAG.	SPECTRUM	LOG E	N	n	M_π	M_s	ΔM	Δ LOG E
Gron. 23	B.D.									
—	$+15°632$.....	3.62	F0	0.14	1	2	0.43			
—	18 640.....	3.63	G9	0.09	1	8	0.44			
28	15 612.....	3.86	G7	0.05	3	17	0.67			
79	17 712.....	3.93	G9	0.05	3	27	0.74			
224	15 631.....	4.04	G9	0.05	4	26	0.85			
143	17 719.....	4.24	A2	0.15	3	18	1.05	0.8	+0.3	−2
170	15 625.....	4.60	A5	0.17	4	27	1.41	1.1	+0.3	−2
315	14 720.....	4.75	A5	0.19	3	15	1.56			
107	17 714.....	4.84	A5	0.19	3	28	1.65			
385	15 666.....	4.85	A3	0.25	2	12	1.66	3.6	−1.9	+5
251	15 637.....	4.84	A5	0.22	2	9	1.65			
38	14 682.....	5.27	F0	0.19	3	23	2.08			
255	15 639.....	5.49	A5	0.20	3	16	2.30			
34	13 663.....	5.59	F0	0.19	3	13	2.40			
87	16 586.....	5.68	A2	0.22	3	18	2.49	1.9	+0.6	−2
245	15 636.....	5.70	F0	0.23	2	11	2.51			
64	13 668.....	5.76	A3	0.25	3	14	2.57	3.6	−1.0	+2
371	15 661.....	5.80	F0	0.19	2	10	2.61			
222	14 702.....	5.97	F0	0.19	3	10	2.78			
282	15 645.....	6.04	F0	0.17	3	16	2.85			
44	13 665.....	6.14	F2	0.11	3	12	2.85			
1	15 603.....	6.35	F5	0.15	2	6	3.16			
150	$+15$ 621.....	6.39	F8	0.12	4	15	3.20			

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[1] *Astronomische Nachrichten*, **211**, 289, 1920.

magnitude to be -0.4. A result of the same order is indicated by the strong absorption in the "cyanogen" bands shown by a spectrogram of the same star, taken by the writer with the Crossley reflector, which places the luminosity near that of β Geminorum ($M=+1.7$). The value $M=+0.69$ found with $\pi=0''.0072$ is in fairly good agreement with these results. The stars from Kapteyn's lists have been combined into groups having the same spectral type and nearly the same values of log E. The absolute magnitude

TABLE II

URSA MAJOR STREAM

Star	Vis. Mag.	Spectrum	log E	N	n	M_π	M_s	ΔM	Δ log E
α Cor. Bor.	2.31	A0	0.12	2	19	0.37	0.4	0.0	0
ϵ Ursae Maj.	1.68	A0*p*	0.15	2	10	0.55	0.8	−0.2	+2
β Serpentis	3.74	A2	0.13	2	13	0.56	0.5	+0.1	0
γ Ursae Maj.	2.54	A0	0.14	3	22	0.60	0.6	0.0	0
78 Virginis	4.93	A2*p*	0.13	2	12	0.66	0.5	+0.2	−1
β Ursae Maj.	2.44	A2	0.15	3	11	0.80	0.8	0.0	0
α Canis Maj.	−1.58	A0	0.20	2	14	1.32	1.5	−0.2	−1
ζ_1 Ursae Maj.	2.40	A0*p*	0.20	2	8	1.44	1.5	−0.1	+1
76 Cygni	6.05	A0	0.19	1	6	1.62	1.4	+0.2	−1
ξ Eridani	5.23	A2	0.20	1	9	1.63	1.5	+0.1	0
δ Ursae Maj.	3.44	A2	0.21	4	29	1.71	1.7	0.0	0
g Ursae Maj.	4.02	A5	0.18	3	18	2.14	.....	......	......
ζ_2 Ursae Maj.	3.96	A0	0.23	1	4	2.25	2.2	0.0	0

was assumed to be the same within each group, and was calculated from the means of Kapteyn's values of the parallaxes.

The fourth column of the tables contains the final values of log E, corrected to $p=0.97$; the fifth and sixth give respectively the number of plates and the total number of images on which these values are based. The seventh column gives the absolute magnitude M_π, derived from apparent magnitude and parallax by the relation $M=m+5+5 \log \pi$. The eighth column gives the absolute magnitude M_s, read by means of log E from the curves in Figure 1. The ninth and tenth columns give the residuals in M and log E from these curves.

The visual apparent magnitudes for the Hyades, the Ursa Major stars, and for Kapteyn's stars are from the Draper catalogue.

TABLE III

PLEIADES, $\pi = 0''.010$

Star		Vis. Mag.	Spectrum	log E	N	n	M_π	M_s	ΔM	Δ log E
Bessel	B.D.	Distance from Alcyone ≷1°								
η	+23°541	2.94	B5*p*	0.06	1	4	−2.06	.....	......	......
f	23 557	3.73	B8	0.06	2	8	−1.27	.....	......	......
b	23 507	3.76	B5*p*	0.11	3	12	−1.24	.....	......	......
c	23 516	3.93	B5	0.08	2	7	−1.07	.....	......	......
e	23 547	4.34	B5	0.09	2	11	−0.66	.....	......	......
d	23 522	4.28	B5	0.06	1	3	−0.72	.....	......	......
h	23 558	5.15	B8*p*	0.11	3	10	+0.15	0.2	0.0	+1
g	23 505	5.53	B5	0.11	1	4	0.53	0.2	+0.3	−2
28	22 563	5.54	B8	0.16	2	9	0.54	0.9	−0.4	+3
m	24 546	5.68	B8	0.09	2	11	0.68	.....	......	−5
k	24 553	5.83	B8	0.12	2	8	0.83	0.4	+0.4	−3
34	23 563	6.23	B8	0.19	3	12	1.23	1.4	−0.2	+1
32	23 561	6.66	B9	0.17	1	4	1.66	1.1	+0.6	−4
38	23 569	6.82	B9	0.24	3	15	1.82	2.5	−0.7	+3
12	24 562	6.88	B9	0.22	3	12	1.88	2.0	−0.1	0
24	23 540	6.93	B9	0.22	3	11	1.93	2.0	−0.1	+1
40	23 570	6.98	A0	0.23	3	17	1.98	2.2	−0.2	+1
29	23 553	6.99	A0	0.22	3	14	1.99	2.0	0.0	0
19	23 537	7.04	A0	0.24	3	16	2.04	2.5	−0.5	+2
22	23 538	7.19	B9	0.25	3	12	2.19	2.8	−0.6	+2
10	23 523	7.37	A0	0.24	3	15	2.37	2.5	−0.1	0
37	23 567	7.44	A0	0.22	3	13	2.44	2.0	+0.4	−2
39	24 578	7.46	A2	0.23	3	11	2.46	2.2	+0.3	−1
31	23 560	7.54	A0	0.28	1	3	2.54	3.6	−1.1	+4
20	24 566	7.68	A2	0.26	2	6	2.68	3.1	−0.4	+1
	24 540	7.79	A3	0.26	2	7	2.79	4.0	−1.2	+3
23	23 539	7.81	A2	0.24	2	5	2.81	2.5	+0.3	−1
7	23 517	7.87	A2	0.27	2	8	2.87	3.3	−0.4	+2
33	23 562	8.01	A2	0.23	2	5	3.01	2.2	+0.8	−3
1	23 510	8.06	A2	0.27	3	6	3.06	3.3	−0.2	+1
	23 496	8.07	A5	0.21	2	7	3.07	.....	......	......
	23 524	8.13	A2	0.29	2	7	3.13	3.9	−0.8	+3
30	23 559	8.14	A3	0.25	2	6	3.14	3.6	−0.5	+1
	23 504	8.22	A5	0.22	2	6	3.22	.....	......	......
	22 573	8.24	A3	0.23	2	6	3.24	2.6	+0.6	−1
	23 495	8.29	A5	0.19	2	9	3.29	.....	......	......
13	23 528	8.46	A5	0.24	2	5	3.46	.....	......	......
	23 508	8.51	A3	0.24	2	4	3.51	3.1	+0.4	−1
	22 565	8.53	A5	0.21	2	6	3.53	.....	......	......
	+24 563	8.59	A5	0.20	2	5	3.59	.....	......	......

TABLE III—*Continued*

STAR Bessel	STAR B.D.	VIS. MAG.	SPECTRUM	LOG E	N	n	M_π	M_s	ΔM	Δ LOG E
		Distance from Alcyone 1° to 3°								
	+21°535	6.0	B9	0.15	3	17	1.0	0.8	+0.2	−1
	22 523	6.7	A0	0.17	3	18	1.7	1.1	+0.6	−4
	22 545	6.9	A0	0.22	3	17	1.9	2.0	−0.1	+1
	22 572	6.9	B9	0.22	3	16	1.9	2.0	−0.1	0
	24 527	7.1	A0	0.21	3	18	2.1	1.7	+0.4	−2
	21 492	7.4	A0	0.20	3	16	2.4	1.6	+0.8	−4
	24 537	7.6	A3	0.23	3	14	2.6	2.6	0.0	0
	20 624	7.7	A2	0.20	3	9	2.7	1.6	+1.1	−5
	22 569	7.8	A2	0.29	3	13	2.8	3.9	−1.1	+4
	25 631	7.8	A2	0.24	3	12	2.8	2.5	+0.3	−1
	22 544	8.0	A2	0.22	3	10	3.0	2.0	+1.0	−4
	24 554	8.2	A3	0.26	2	8	3.2	4.0	−0.8	+2
	24 568	8.4	A3	0.21	2	8	3.4	1.9	+1.5	−4
	22 556	8.5	A2	0.27	2	7	3.5	3.4	+0.1	−1
	21 530	8.3	F0	0.13	2	6	3.3	.....	......	......
	24 528	8.9	A5	0.22	2	4	3.9	.....	......	......
	+25 615	8.9	A3	0.25	2	4	3.9	3.6	+0.3	−1

TABLE IV

PRAESEPE, $\pi = 0''.0072$

Star B.D.	Vis. Mag.	Spectrum	log E	N	n	M_π	M_s	ΔM	Δ log E
+20°2171	6.23	A2	0.17	2	14	0.52	1.1	−0.6	+4
20 2149	6.39	F0	0.15	2	10	0.68	.....	......	......
20 2166	6.40	G5	0.05	2	2	0.69	.....	......	......
20 2158	6.48	K0	0.08	2	5	0.77	.....	......	......
20 2159	6.48	A0	0.11	2	6	0.77	0.2	+0.6	−4
19 2069	6.54	F0	0.14	2	9	0.83	.....	......	......
19 2053	6.58	A5	0.13	2	13	0.87	.....	......	......
20 2175	6.60	A3	0.17	2	10	0.89	1.1	−0.2	+1
20 2172	6.77	A5	0.19	2	9	1.06	.....	......	......
20 2185	6.98	K0	0.10	2	3	1.37	.....	......	......
20 2143	7.09	F0	0.20	2	5	1.38	.....	......	......
20 2152	7.26	A0	0.16	2	8	1.55	0.9	+0.7	−4
20 2163	7.41	A0	0.24	1	4	1.70	2.5	−0.8	+3
20 2153	7.52	A0	0.18	2	6	1.81	1.2	+0.6	−3
19 2064	7.54	A3	0.17	2	7	1.83	1.1	+0.7	−4
20 2148	7.57	A3	0.21	2	6	1.86	1.9	+0.0	0
20 2132	7.72	A0	0.26	2	6	2.01	3.1	−1.1	+4
20 2165	7.74	A0	0.20	2	6	2.03	1.6	+0.4	−2
19 2078	7.77	A3	0.21	2	5	2.06	1.9	+0.2	−1
+19 2084	7.88	A5	0.20	2	6	2.17	.....	......	......

Those of the Pleiades are by Miss Parsons,[1] or by Müller and Kempf with the reduction −0.30 to the Harvard scale. For a few Pleiades stars Münch's photographic magnitudes have been used, reduced to the visual scale by means of color-indices inferred from the spectral type, which is not far from A0. The first part of Table III contains members of the group within 1°; the second, between 1° and 3° from Alcyone. For the latter stars the photographic magnitudes derived by Trumpler have been reduced to the visual scale in the same way as those by Münch. For Praesepe the photographic magnitudes by Kohlschütter on the Göttingen system were similarly reduced to visual magnitudes. The spectra are given according to Harvard with the exceptoin that for Pleiades stars fainter than $7^{\mathrm{M}}.5$ the spectral types have been revised on the basis of the strength of the K line.

TABLE V

KAPTEYN STARS, COMP. o_2 ERIDANI, α CYGNI

Star	Vis. Mag.	Spectrum	log E	Mean	N	n	M_π	M_s	ΔM	$\Delta \log E$
Boss 4018......	4.11	B9	0.11	......	1	7	......	....	.....	.....
1438......	4.92	B9	0.10	......	2	12	......	....	.....	.....
1455......	5.57	B9	0.11	0.12	3	19	0.01	0.4	−0.4	+2
1448......	6.61	B9	0.16	......	3	11	......	....	.....	.....
Boss 1944......	3.09	B8	0.08	......	1	5	......	....	.....	.....
4465......	4.65	B8	0.08	......	1	5	......	....	.....	.....
2071......	5.11	B8	0.08	......	1	5	......	....	.....	.....
4058......	5.41	B8	0.07	0.08	1	5	−0.16	....	.....	.....
4006......	5.61	B8	0.07	......	1	6	......	....	.....	.....
4573......	5.79	B8	0.06	......	1	5	......	....	.....	.....
1788......	5.88	B8	0.08	......	1	6	......	....	.....	.....
Boss 1262......	3.68	B5	0.04	0.05	1	8	......	....	.....	.....
926......	5.25	B5	0.06	......	1	7	−1.27	....	.....	.....
Comp. o_2 Eridani	9.5	B9	0.31	......	2	2	10.3	....	.....	.....
α Cygni........	1.33	A2p	0.08	......	1	8	......	....	.....	.....

Table VI contains the results of measurements on later-type stars from the list of spectroscopic parallaxes by Adams, Joy, Strömberg, and Burwell, made in connection with an investigation of the "cyanogen" absorption beyond λ 3889. The means were taken for limited intervals of spectral type and absolute magnitude; *N* is the number of stars in each group. No correction was made for a variation of Schwarzschild's constant, and the measurements for the individual stars are not as accurate as for the A-type stars in the preceding tables. In fact, with the decrease in intensity of Hζ and the general strengthening of the metallic lines in later spectral types the region λ 3895–λ 3907 has no longer the same uniform appearance as for the early types. The results in the table show, however, that the effect still persists to some extent in the later types, in spite of the vanishing of Hζ.

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[1] *Astrophysical Journal*, 47, 38, 1918.

TABLE VI

MEAN RESULTS FOR STARS WITH DETERMINED SPECTROSCOPIC PARALLAXES

Spectral Range	Mean Type	*N*	Mean Abs. Mag.	Mean log *E*
F5*p*–G3*p*..................	G0*p*	4	−2.5	0.05
F4 –G8..................	G1	3	−1.7	0.04
A8 –F9..................	F4	3	0.3	0.09
F1 –G1..................	F7	15	3.9	0.15
G1 –K2..................	G8	12	0.9	0.05
G4 –K2..................	G9	6	5.7	0.16
K3 –Ma..................	K5	7	0.6	0.12
K3 –K7..................	K5	3	7.3	0.24

Figures 1 and 2 illustrate the result in the tables, log *E* being plotted against M_r, with different symbols for the different groups. *α* Cygni and Comp. o_2 Eridani are represented by arrows. The absolute magnitude of *α* Cygni can only be estimated qualitatively as very bright from exceedingly small proper motion. Comp. o_2 Eridani with $M = 10.3$ falls below the margin of the diagram. Figure 1 gives the results for spectral types B5–A3; Figure 2 for types from A5 on. From B8 to A2 there is no certain evidence of a change in the curve; but the A3 curve for lower luminosities evidently falls below the B8–A2 curve, and forms a transition to the A5 curve in Figure 2. The degeneration of the curves for types A5–F is clearly a consequence of the decreasing intensity of Hζ with advancing type.

It is evident from these results that the increase of log *E* with decreasing luminosity can be used for determining absolute magnitudes for stars of spectral types B8–A3, but that the curves for later types cannot give very trustworthy results. The curves further show that for very bright stars log E approaches a limiting value representing a very smooth run of the energy-curve between

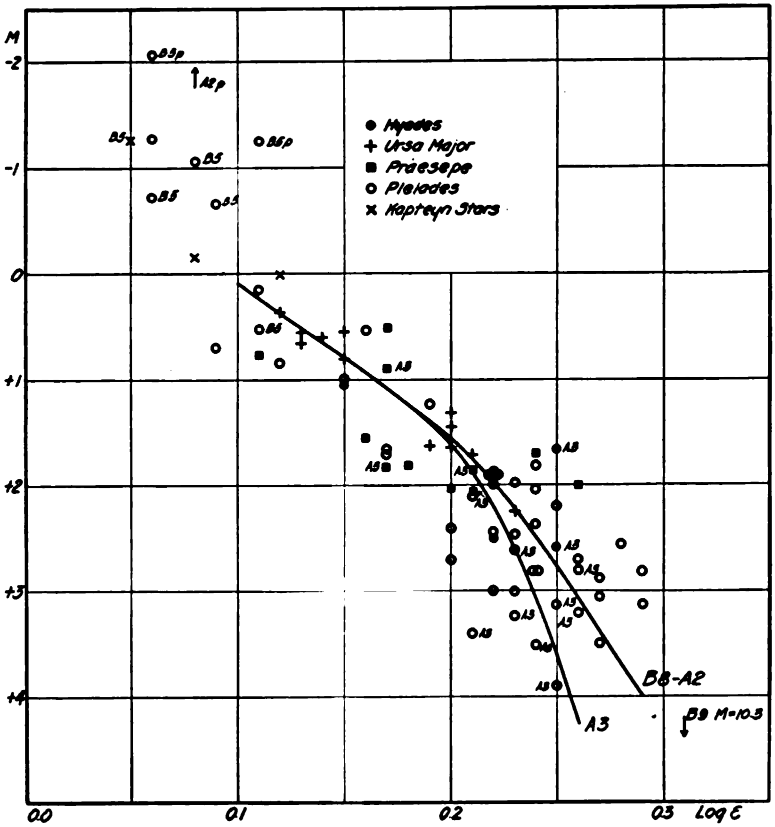

FIG. 1.—Logarithmic exposure-ratio for the regions λλ 3895–3907, 3907–3925, and absolute magnitude. Spectral types B5–A3.

Hζ and K, a consequence of the extreme narrowness of all absorption lines. This phenomenon seems to be more or less independent of type, *a* Cygni, A2*p*, falling in with the B5 stars of the Pleiades. The Cepheids and Pseudo-Cepheids and the very bright F4–G8 stars in Table VI also fall near the same point of the diagram. The steepness of the curve for stars brighter than $M = 0$ is such that only

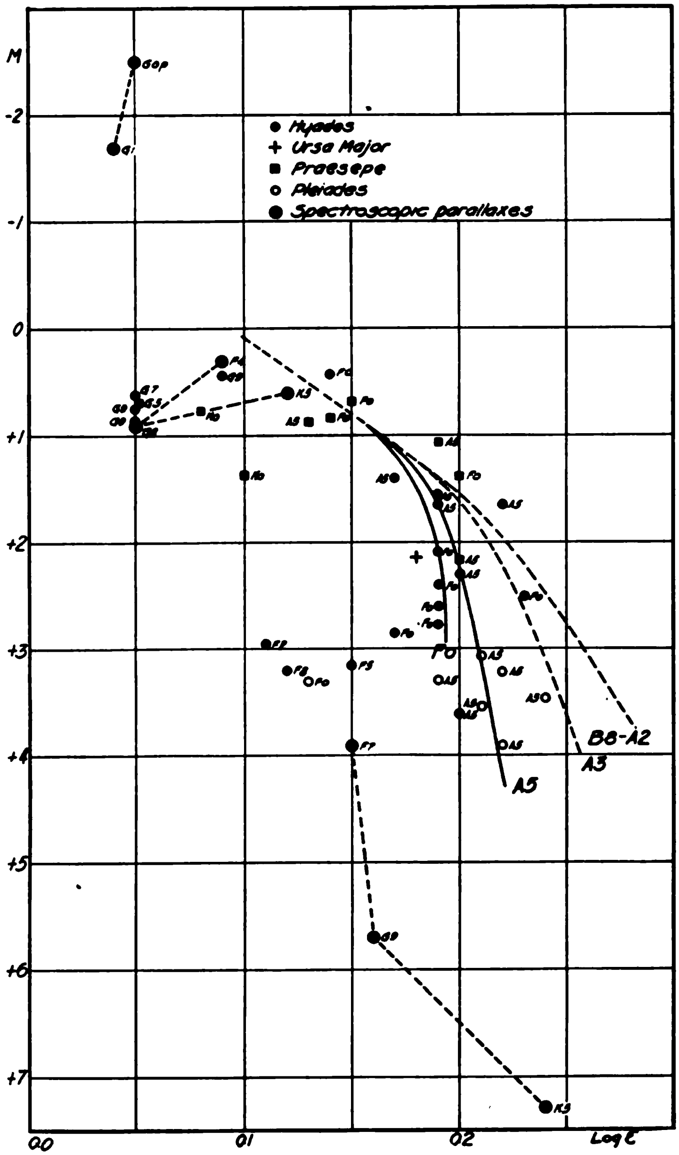

FIG. 2.—Logarithmic exposure-ratio for the regions λλ 3895–3907, 3907–3925, and absolute magnitude. Spectral types later than A5.

qualitative estimates of the luminosity can be made from values of log E less than 0.10.

To form the B8–A2 curve, means of log E for certain intervals of M were taken for both Ursa Major and Pleiades stars. Between $M=0$ and $M=+2$ the curve was drawn to represent the Ursa Major stars. A comparison with the eleven Pleiades stars in this region observed on three plates gave a systematic difference of +0.002 in log E, which is negligible. The remaining parts of the curves were then drawn in accordance with the results for the Pleiades, attention being given also to the Hyades and Praesepe stars in forming the A3 and A5 curves. The co-ordinates of the curves in Figure 1 then run as follows:

	B8–A2				
log E	0.10	0.15	0.20	0.25	0.28
M	+0.08	0.80	1.55	2.77	3.68
	A3				
log E	0.10	0.15	0.20	0.22	0.25
M	+0.08	0.80	1.62	2.20	3.60

The F0 curve in Figure 2 is drawn mainly from stars in the Hyades.

The absolute magnitudes corresponding to log E, M_s, have been read from these curves for all B8–A3 stars having log $E>0.10$; the two A5 stars in Figure 2 with log $E<0.18$ are included.

Errors in the assumed values of the parallaxes evidently combine with residual systematic errors in the measurements. From the discussion of systematic errors it seems that the reliability of any correction to the parallax that might result from the mean residuals for a group will probably depend more on the number of images measured for the individual stars than on the number of stars. The absolute magnitudes of the Ursa Major stars derived from stream-motion are probably but little affected by systematic errors. This likely is also true of the values of log E, because these stars were observed individually on a rather large number of different plates. As there seems to be no appreciable systematic difference between the Ursa Major stars and the Pleiades we may conclude that Trumpler's value of the parallax, $\pi = 0\overset{''}{.}010$, is very good. With allowance for a possible systematic error of about 0.01 in log E the error may amount to about 10 per cent, or $\pm 0\overset{''}{.}001$. For the Hyades and Praesepe we can determine the parallax which causes the systematic difference in log E to vanish. For the Hyades we find $\pi = 0\overset{''}{.}023$, the value initially adopted; and for Praesepe, $\pi = 0\overset{''}{.}0068$, differing by only $0\overset{''}{.}0004$ from Kohlschütter's value. An uncertainty of 10 per cent is probably to be admitted for both these values. The Kapteyn stars of types B8 and B9 seem to agree with the curve as well as can be expected.

The residuals ΔM for the individual stars are seen to be very small for the Ursa Major stars, but are rather large for some of the A3 stars of low luminosity in the other groups. This is to be expected from the steepness of the curve. In all, the residuals for seventy-one stars give a probable deviation of $\pm 0\overset{M}{.}40$, which is comparable with the probable error of the spectroscopic method for late-type stars. With weights equal to the number of images, n, the probable deviation in log E from the curves is found to be ± 0.026 for a value based on four measured images. This value is to be compared with ± 0.025 previously derived from the residuals in log E from the mean for different plates. The latter value, it was concluded, was produced to a considerable extent by systematic errors common to a large number of stars in a certain group. In the new value, on the other hand, the systematic errors are combined with errors in the parallaxes, and may be partly eliminated in drawing the curves. The close agreement of the two results suggests, however, that the correlation between the energy distribution around λ 3907 and the absolute magnitude is close.

An investigation of spectra of higher dispersion for some selected A stars of different luminosities, photographed with a focal-plane spectrograph in the Newtonian focus of the 60-inch reflector, and with the Cassegrain spectrograph and an 18-inch camera, and also with a slitless quartz spectrograph used in conjunction with the Crossley reflector of the Lick Observatory, throws some light on the question of the physical cause of the effect investigated here. Undoubtedly the increase of log E with decreasing luminosity is to a considerable degree caused by a widening of the wings of the Hζ line, the extreme limits being represented by very bright stars such as α Cygni and α Leonis with narrow hydrogen lines and the very faint companion of o_2 Eridani, which has exceedingly wide lines. But there seems also to be evidence of another cause. It appears as though certain arc lines of iron and silicon in the region λ 3889–λ 3907 increase appreciably in width and intensity with decreasing luminosity. The lines suspected are λλ 3895.8, 3899.9, 3903.1, 3906.6 of iron, and λ 3905.6 of silicon. With the exception of λλ 3903.1 and 3905.6, these lines are classified by King[1] as belonging to Class IB. The exceptions are in classes II and III, respectively. A widening of these lines with decreasing luminosity seems to contribute perceptibly to the depression of the intensity in the region Hζ–λ 3907. There seems to be a similar effect, though much weaker, in the region λ 3914–λ 3923, including the two iron lines λλ 3920.4 and 3923.1 of Class IB. On the whole, in stellar spectra the region between λ 3907 and K is certainly affected by heavy arc lines to a much smaller degree than the region Hζ–λ 3907.

In some spectra of Sirius photographed by Nicholson with the Snow telescope, with a dispersion of 3 A per mm, the lines mentioned between λλ 3889 and 3907 are plainly visible, except the iron line λ 3906.6. The line λ 3905.6 is relatively strong, of about the same strength as the enhanced titanium lines λλ 3900.7 and 3913.7. We probably have here a blend of the silicon line with an enhanced line of chromium, but an indication that the blend is to a considerable part due to silicon may be found in the circumstance that the Kensington observers[2] state that it is somewhat weaker in the spectrum of ε Ursae Majoris than in the spectrum of Sirius, in spite of the fact that of thirty enhanced chromium lines all are found stronger in the former star, except one which is given as of the same intensity in both spectra. In the case of α Lyrae and α Coronae, which are stars of A0 type more luminous than Sirius, λ 3905.6 is still to be seen in the spectrograms taken with the Cassegrain spectrograph; of the iron lines only λ 3903.1 can be traced with certainty. On the other hand, in ζ_2 Ursae Majoris with $M = +2.25$, λ 3903.1 is strong, and λλ 3895.8 and 3899.9 are clearly visible; in the position of λ 3906.6 there is a very broad line seemingly including λ 3905.6 as a blend. The prominent arc lines of iron between Hζ and Hη are very broad and well marked, probably affecting to a perceptible extent the integrated intensity of the region between these two lines, at least from λ 3860 to Hη. To the red of λ 3907 no effects of similar importance for the variation of the integrated intensity of wide patches of the continuous spectrum can be seen, though the spectrum is evidently rich in lines. The K line, which indicates that the spectral type lies between Ao and A2, is weaker and sharper than in the spectrum of β Serpentis, type A2, $M = 0.56$, in which the iron arc lines are faint. The visibility of the broad arc lines is therefore not likely to be interpreted as due to an increase in spectral type. In addition to these comparisons made with the Cassegrain spectrograph, the slitless spectrograms taken with the quartz spectrograph on the Crossley reflector, and the low dispersion slit spectra obtained with the 60-inch reflector, show an abrupt fall of intensity near λ 3907 for ζ_2 Ursae Majoris, as well as for Nos. 10, 20, and 31 of the Pleiades, types Ao, A2, Ao, $M = +2.37$, 2.68, 2.54, respectively, for No. 64 of the Hyades, A3, $M = +2.57$, and the Praesepe star B.D. $+20°2165$, Ao, $M = +2.03$. The difference relative to the brighter Pleiades and Ursa Major stars is plainly recognizable. A very marked depression near λ 3906 is shown on a spectrogram of Comp. o_2 Eridani taken with the quartz spectrograph. From the appearance of the K line this star ought probably to be classified a little earlier than Ao.

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[1] *Mt. Wilson Contr.*, No. 66; *Astrophysical Journal*, 37, 239, 1913; *Publications of the Astronomical Society of the Pacific*, 33, 106, 1921.

[2] Solar Physics Committee, *Chemical Origin of Various Lines in Solar and Stellar Spectra*, p. 49, 1910.

The effect in question persists for later types, as shown by Table VI and is confirmed by results from spectra of other kinds. The energy-fall in a slitless spectrogram at λ 3907 is much more pronounced for a dwarf than for a giant. This fact confirms our results just derived, because the importance of the hydrogen line Hζ for the effect must be negligible in these types. On the other hand, the degeneration of the effect toward type F shows that Hζ probably causes a large part of the phenomenon in the A type.

The facts dealt with here are probably best understood from the standpoint of an increased density-gradient in the outer layers of a star of certain type with decreasing luminosity, which follows because the change in luminosity is in all probability due chiefly to a change of the mean density. For a layer of certain temperature the density will be appreciably greater in a star of low luminosity than in a more luminous one, a condition which may be thought to favor the strength and width of the arc and flame lines.

The eclipsing binaries offer an opportunity to derive a relation between the effect at λ 3907 and the mean density of the star. From some preliminary results obtained here it has been found that the energy-fall near λ 3907 is strong for RZ Cassiopeiae and RR Draconis, whose brighter components have especially high mean densities, and seems to be much smoother for the low-density stars examined. For RZ Cassiopeiae a value of M between $+2$ and $+3$ inferred from the spectrum is in fair agreement with Dugan's[1] results for the dimensions of the brighter component of this star.

II. THE "CYANOGEN" ABSORPTION IN SPECTRA OF TYPES G–M

Miss Maury[2] makes the statement in her classification of stellar spectra that the sudden decrease of general intensity in the spectrum from λ 3889 toward the ultra-violet found in types XIV and XV, as well as the degree of absorption in the regions λ 4055–λ 4078 and λ 4144–λ 4216, is developed to a different degree for different stars within these types. The variation of intensity in these regions is further thought to be correlated to some extent with the strength of the lines $\lambda\lambda$ 4215.7, 4227, and the compound line λ 4076.8–λ 4077.9. Kapteyn[3] showed that stars with heavy absorption in the violet and ultra-violet regions mentioned have on the average much smaller proper motions than stars with comparatively slight absorption. This result was thought by him to bear upon the problem of selective absorption in inter-stellar space. In more recent years, however, the influence of differences in the absolute magnitudes of the stars on the appearance of their spectra has been recognized, especially by Adams and Kohlschütter in the development of the spectroscopic method for deriving stellar parallaxes. In this method the lines λ 4216 and λ 4078 are two of the most important variable lines. That the energy-fall beyond λ 3889 is intimately connected with the spectroscopically determined absolute magnitude was found by the writer by using very short objective-grating spectra.[1] The color-effect of the absolute magnitude was found to be much more strongly developed at the inner limit of the spectra in the "minimum wave-length" than in the "effective wave-length" measured by bisecting the images. It was then suspected that the effect could be due to an increase of absorption with stellar luminosity in the heavily winged lines of iron in the region λ 3870–λ 3890.

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[1] *Contributions from the Princeton Observatory*, No. 4, 1916.

[2] *Harvard Annals*, **28**, 38, 1897.

[3] *Mt. Wilson Contr.*, No. 31; *Astrophysical Journal*, **29**, 46, 1909.

It has been found recently, however, that the effect is without doubt due to a variation in strength with stellar luminosity in the strong "cyanogen" band with its first head at λ 3883. On spectra of stars of different types and luminosities photographed with a slitless quartz spectrograph[2] on the Crossley reflector of the Lick Observatory three similarly behaving absorption regions were found to coincide with the three "cyanogen" bands with first heads at λλ 4216, 3883, and 3590. The band at λ 3883 is by far the strongest; the point most sensitive to changes in luminosity seems to be the region around and between the two heads at λ 3871 and λ 3883. Of the two other bands, that at λ 3590 seems to be somewhat stronger. The band at λ 4606 seems to be too faint to be identified with certainty.

The spectra of the following stars were photographed; the spectral types and absolute magnitudes are from the new list of spectroscopic parallaxes[3] by Adams and his collaborators.

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[1] *Astrophysical Journal*, **49**, 289, 1919; *Upsala Universitets Årsskrift*, 1920.

[2] This instrument is described by W. H. Wright in the *Lick Observatory Bulletin*, **9**, 52, 1917.

[3] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

χ Leonis	F5,	0.8
39 Leonis	F8,	3.7
Lal 19896	G1,	4.5
Boss 2338	G3,	−1.0
Lal 21008	G6,	5.2
ε Virginis	G6,	0.0
ε Geminorum	G8,	−1.4
β Geminorum	G9,	1.7
θ_1 Tauri	G9,	1.4
o_2 Eridani	K1,	6.2
Cin. 1712	K3,	6.4
α Tauri	K4,	1.0
β Cancri	K4,	0.4
Lal 13284	K5,	7.1
Boss 2935	Ma,	10.5
Boss 1868	Ma,	−0.2
α Orionis	Ma,	−3.8

The development of the bands is at a maximum for giant stars of types between G5 and K5, and diminishes strongly toward the F and M types. In the dwarf stars of all types the bands are only faintly developed.

The absorption in the band at λ 3883 has been investigated here for stars of known spectroscopic parallax by means of the 6° objective prism on the 10-inch refractor. The method employed is analogous to that used for the A stars described in the first part of this paper. This investigation was carried out before the identification with the "cyanogen" band had been made. The two regions of the spectrum immediately adjacent to λ 3889 were compared in series of images taken with decreasing exposure-time to determine the exposure-ratio for equal photographic density in the two regions. The maximum effect seems to be in spectral types G8–K0, where the difference in exposure-ratio between average giants and dwarfs corresponds to nearly a magnitude. The effect decreases rapidly from G5 to G0. The Cepheids and pseudo-Cepheids have a somewhat peculiar position with even less contrast at λ 3889 than dwarfs of the same spectral type. The smoothness of the spectrum around the $H\zeta$ line for these stars is probably due to the disappearance of the wings of the strong absorption lines following the advance of the spectrum toward Miss Maury's *c*-stage.

There is a very decided disadvantage, however, in the fact that the region in question is situated so far in the violet that for the redder stars the exposures must be made considerably longer. The band at λ 4216, though weaker, is much more favorable in this particular, and an investigation of it was therefore begun with the same instrumental arrangement. The region most strongly affected by this band lies between λλ 4144 and 4184.

This region of 40 A width, corresponding to about 0.09 mm on the plate, has been compared with that at λ 4227–λ 4272 as standard. The regions in question are easily identified with sufficient accuracy by estimating the distances from the G-band and the line λ 4227, which are conspicuous features of the spectra even with very low dispersion. For the dwarfs of types K5–M the intensity in the second region mentioned is probably somewhat modified by the wings of λ 4227 which produce an additional effect of luminosity on the exposure-ratio in the same direction as the effect of the "cyanogen" absorption in the first region. No special efforts were made to determine Schwarzschild's constant for the plates. The results obtained from the investigation of the A stars indicate that generally the changes in the constant are small and do not increase materially the probable error of the measured exposure-ratio, but that rather large changes may occur, especially from one emulsion to another. In this case several stars were measured on each plate, and systematic errors of large size ought to show in the residuals from the curves. The means of the residuals for the individual plates are found to be small, however.

As before, the plates used were Seed 23, size 8×10 inches, with X-ray developer. Images more than 3.6 inches from the center were excluded. The focus was set near λ 4200, and is tolerably constant over most of this area. Even within the adopted limit of distance, however, some images must be excluded by reason of distortion or bad focus, because the field has not exact symmetry of rotation with reference to the plate center.

A scheme for classifying the objective-prism spectra was developed by using types determined by Adams and his collaborators as standards. The classification is based on the appearance and relative strength of the hydrogen lines, the K line of calcium, the λ 4227 line of calcium, the titanium-oxide bands, and the general energy-distribution in the spectrum. Since the decrease of the hydrogen lines and the increase of λ 4227 with advancing spectral type proceed more rapidly in the dwarf series, while the increase of redness takes place faster in the giant series, and since λ 4227 has enormous strength in dwarfs of types K5–M, these characteristics

TABLE VII

Boss or Cin. No.	Harv. Vis. Mag.	Spectrum	*M*	log *E*	*n*	Spectrum Obj. Prism
3274	6.5	G8	0.4	0.25	6	G8*g*
3298	4.8	K3	0.6	0.27	8	K0*g*
3326	5.9	K0	4.2	0.16	4	K2
3344	6.5	G8	0.9	0.26	7	G8*g*
3362	4.9	Ma	−0.3	0.18	8	Ma*g*
C 1657	7.7	K2	6.4	0.14	4	K0*d*
3408	6.2	K4	0.5	0.25	4	K5*g*
3537	5.7	K3	0.4	0.25	4	K2*g*
3554	6.3	F9	4.3	0.12	6	G5*d*
4131	5.5	K2	1.1	0.27	5	K2*g*
4137	5.6	G2	4.7	0.12	7	G5*d*
4147	3.3	G6	0.0	0.19	5	G5*g*
4257	6.7	G8	4.7	0.13	7	K0*d*
4264	5.4	K5	0.8	0.24	7	K5*g*
4315	3.4	K2	0.6	0.26	5	K0*g*
4318	5.9	Ma	0.2	0.22	4	Ma*g*
4323	5.0	K5	0.2	0.29	5	K2*g*
4327	4.4	G2	−1.4	0.27	4	G5*g*
4342	7.9	K5	7.9	0.05	6	K5*d*
4370	5.3 5.3	K1 K2	6.3 6.3	0.12	7	K0*d*
4372	6.7	K4	7.3	0.08	5	K5*d*
4414	6.3	K1	1.3	0.24	3	K2*g*
4420	4.3	A9	1.6	0.12	4	F2
C 2322	7.9	K5	8.0	−0.02	4	K5*d*
Lal 31905	7.2	K2	2.1	0.25	4	K5*g*
4571	4.3 5.0	K0 K4	5.8 7.8	0.10	6	G8*d*
4580	4.7	G6	−0.1	0.27	6	G8*g*
4595	5.7	F4	2.8	0.12	6	F5
4618	6.1	G2	−0.3	0.20	6	G5*g*
4635	4.9	G5	−0.5	0.20	9	G0*g*
4638	3.4	G8	2.5	0.15	5	G8
A. G. Leip. 8449	7.6	K2	0.8	0.33	5	K2*g*
4660	5.7	G0	−0.8	0.22	9	G0*g*
C 2420	7.9	G5	5.5	0.11	5	G5*d*
C 2421	8.5	G9	5.2	0.10	6	G5*d*
A. G. Leip. 8507	8.9	G2	5.7	0.14	5	G5*d*
4678	5.4	G9	1.7	0.21	5	K0*g*
A. G. Leip. 8585	7.6	Ma	1.5	0.16	3	Ma*g*
4688	6.5	G8	2.8	0.22	3	K0*g*
4713	5.4	F0	1.6	0.11	6	A–F

TABLE VII—*Continued*

Boss or Cin. No.	Harv. Vis. Mag.	Spectrum	*M*	log *E*	*n*	Spectrum Obj. Prism
C 2456	8.8	Mb	10.7	−0.09	3	Mb*d*
4800	4.5	Mb	−1.2	0.15	8	Mb*g*
4817	5.4	F8	0.3	0.13	5	F8
4823	4.2	K1	1.0	0.28	11	K0*g*
C 2495	6.0	G4	3.7	0.10	10	G5*d*
4892	6.8	G4	5.5	0.13	3	G2*d*
4893	6.6	G6	5.7	0.12	3	G2*d*
4898	5.7	G8	−1.4	0.28	12	G8*g*
4910	6.3	G8	1.5	0.24	2	G5*g*
4912	4.5	K0	0.2	0.29	8	K2*g*
4920	6.0	F4	2.8	0.12	3	F8
4940	4.6	K4	0.5	0.26	4	K3*g*
4950	5.2	G8	5.1	0.13	5	G5*d*
5009	4.8	G9	5.8	0.09	6	G8*d*
5027	4.4	G8	0.0	0.22	6	G8*g*
5052	3.8	Ma	0.2	0.11	5	Ma*g*
5079	4.0	G2	0.7	0.13	6	G5*g*
5089	4.9	G9	0.8	0.18	5	G8*g*
5093	3.9	G8	3.9	0.13	5	G8
5098	5.8	G1*p*	−2.5	0.20	4	G5*pg*
5118	3.7	K6	0.5	0.25	6	K5*g*
5125	5.6	Mc	0.5	0.14	5	Mb*g*
5129	4.6	Mb	0.3	0.13	8	Mb*g*
C 2616	7.2	G8	5.6	0.13	2	G8*d*
5143	5.6	G8	0.0	0.27	2	K0*g*
5180	5.7	K2	5.9	0.13	6	K2*d*
5187	4.0	G9*p*	−2.0	0.23	4	G*pg*
5219	6.1	G2	5.2	0.06	3	G5*d*
5229	2.3	G0*p*	−3.0	0.24	5	G*pg*
C 2672	6.3	G8	6.2	0.12	4	G8*d*
5321	6.0	F5	3.6	0.13	2	F5
5346	3.6	G8	3.2	0.18	5	G8*g*
C 2707	8.6	Ma	9.4	−0.01	3	M*d*
C 2718	6.6	F7	3.9	0.10	9	F5
5420	6.2	G8	1.5	0.26	7	K0*g*
5431	3.9	K2*p*	−2.9	0.24	4	K*pg*
5433	5.6	K7	8.3	0.09	9	K5*d*
5434	6.3	K8	8.7	0.05	7	Ma*d*

alone will suffice in a great number of cases to determine the spectral type and to decide whether the star is a giant or a dwarf. From type G0 on, however, the degree of the "cyanogen" absorption must be taken into account, and the approximate luminosity will be obtained with the aid of log E. The spectra used are narrow and, in what concerns the visibility of the lines, are inferior to widened images, but this has been at least partly compensated by the large number of images with different intensities usually available. For spectral types G0–K0 both the hydrogen lines and λ 4227 are only faintly visible, but a fairly good discrimination between the stages G0, G5, and G8–K0 seems to be possible. Four Cepheids and pseudo-Cepheids among the stars observed were recognized as peculiar stars mainly by the abnormal strength of the hydrogen lines.

The results of the measurements and classifications are given in Table VII. The first four columns contain data from the list of spectroscopic parallaxes. The fifth column gives log E, the logarithmic exposure-ratio for equal intensities in the two regions λ 4144–λ 4184 and λ 4227–λ 4272, the former being situated in the densest part of the "cyanogen" band; n is the number of images used for determining log E. The last column gives the spectral type as determined from the objective-prism spectra, in most cases without knowledge of the more accurate values of the type in the third column. An added g or d indicates a giant or a dwarf, respectively. This estimation of luminosity is made mainly from the value of log E according to the curves in Figures 3 and 4, the value of the spectral type estimated from the objective-prism spectra being used as argument. The individual results given in the last column are therefore directly dependent only on the data secured from the low dispersion spectra. The good agreement between these results and the spectral type and absolute magnitude in the third and fourth columns seems to give promise of the usefulness of the method.

Figure 3 gives the relation between log E and the absolute magnitude for stars of types A9 to K2, Figure 4 for types K2–Mc. The curves drawn representing the different types have only provisional significance. Figure 5 gives log E as a function of the spectral type, different degrees of luminosity being represented by different notations. It is evident from these diagrams that the effect of the "cyanogen" band reaches a maximum near Type K2 and decreases toward G0 and M. The analogy between

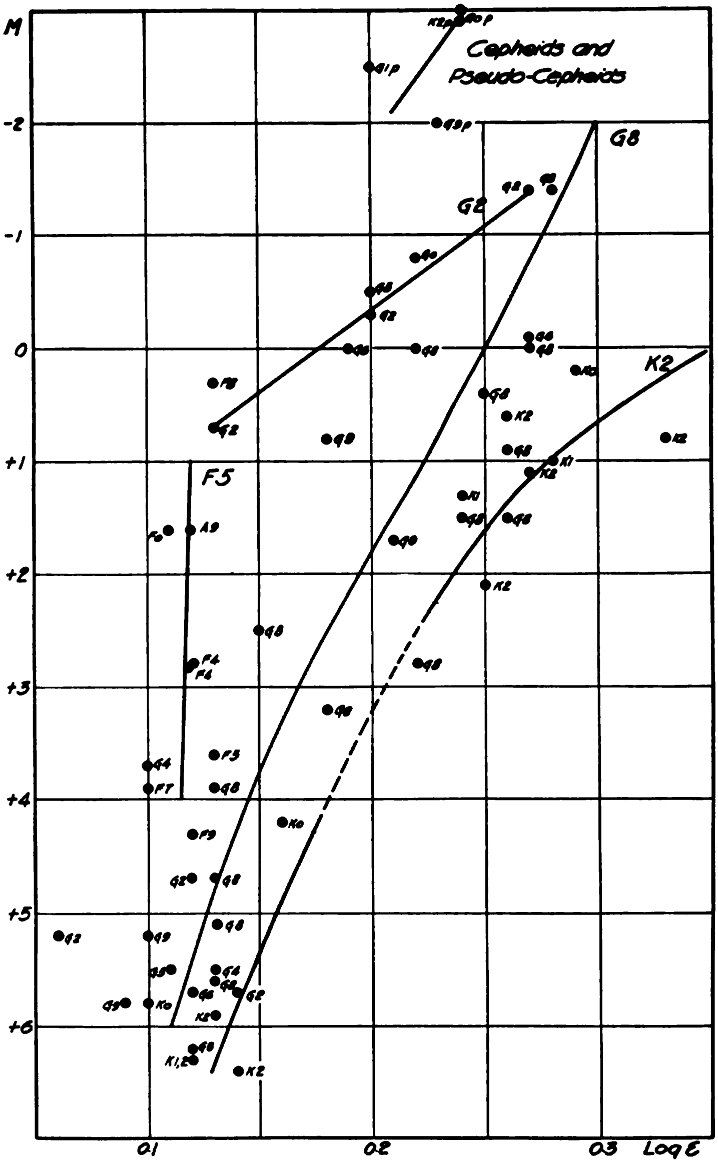

FIG. 3.—Logarithmic exposure-ratio for the regions λλ 4144–4184, 4227–4272, and absolute magnitude. Spectral types A9–K2.

Figure 5 and the diagram for the minimum wave-length of the grating-spectra[1] is very striking. The advantage of the present diagram, however, lies in the circumstance that the effect of the regular change in the energy-distribution in the spectrum due to the change of temperature in the spectral series has practically no effect on log E, owing to the closeness of the two compared regions. Except for the position of the Cepheids and pseudo-Cepheids the

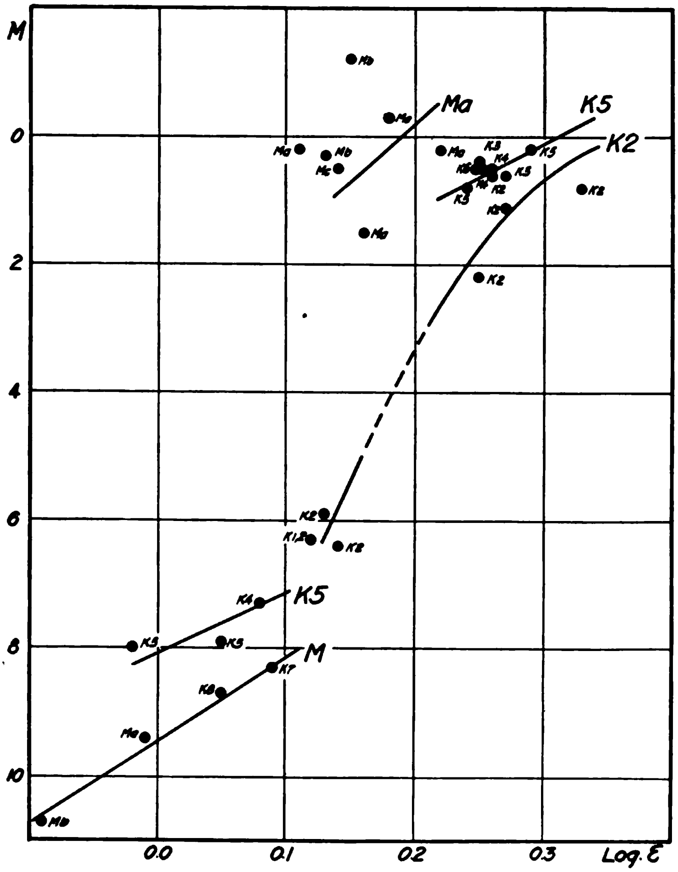

Fig. 4.—Logarithmic exposure-ratio for the regions λλ 4144–4184, 4227–4272, and absolute magnitude. Spectral types K2–Mc.

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[1] *Loc. cit.*

correspondence is nearly perfect between Figure 5 and a similarly arranged diagram for the absorption in the λ 3883 band. The variation in log E for the stronger ultra-violet band is, however, about 2.3 times as large as for the λ 4216 band. The increase of the effect from G8 to K2 probably explains the correlation of the absorption in the bands with the line λ 4227 found by Miss Maury in type XV.

The rapid decrease of the effect toward G0, especially for stars with M between 0 and $+1$, is a disadvantage for a more accurate determination of luminosity. On the other hand, stars of this brightness, between types F0 and G2, seem to be very scarce. Most of the giants in this interval of spectral type given in the list of 1646 spectroscopic parallaxes, for example, are Cepheids or pseudo-Cepheids and of much higher luminosity.

The Cepheid variables and the pseudo-Cepheids seem to have an abnormally small absorption in the regions of the "cyanogen" bands; it is somewhat stronger, however, in the region of the λ 4216 band than in that of the λ 3883 band. In the case of the former band the stars still fall among the ordinary giants in the diagram in Figure 5, but in the case of the ultra-violet band, as mentioned before, they lie even below the dwarfs.

From K2 to M there is a general decrease of log E; the difference between giants and dwarfs is still well accentuated. The extent to which this difference is still due to "cyanogen" absorption in type M seems to be uncertain. For this type the line λ 4227, strong in the dwarfs, and the hydrogen lines, which are strong in the giants,[1] give a reliable criterion of the order of magnitude of the luminosity. The titanium-oxide bands characterizing type M are generally well developed in the objective-prism spectra. An observation of importance to be made from Figure 5 is that stars with log $E > 0.17$ are giants.

An application of the preceding principles to stars of faint apparent magnitudes in a few rich regions has been made with a small focal-plane spectrograph attached to the 60-inch reflector. The spectrograph has a 64° U.V. prism; the focal lengths of collimator and camera are about 6 inches. The dispersion gives 1.51

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[1] See various papers by Adams and his associates on the spectroscopic method.

mm between Hγ and Hδ, and is 2.8 times that of the objective-prism spectra. The slit was removed, thus giving a field about 9′×9′, of which about two-thirds is in good enough focus to be used. The objects investigated were the clusters M 11 and 13, and a few fields in Selected Areas Nos. 63 and 64 in the Cygnus region. A list of the photographs is given in Table VIII. Seed 30 plates were used, except for No. 33, which is Seed 23. All the plates were taken during good seeing, estimated about 5 and 6 on a scale of 10 in most cases. The last column gives for the Selected Area plates the number of the star in the Harvard Durchmusterung[1] which is nearest to the center of the field.

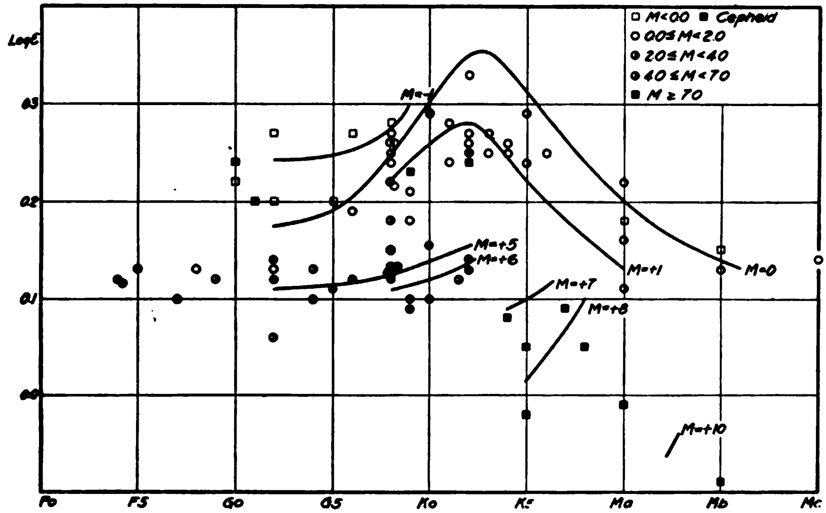

FIG. 5.—Logarithmic exposure-ratio for the regions λλ 4144–4184, 4227–4272, and spectral type.

Messier 11, R. A. 18ʰ46ᵐ, Decl. −6°23′ (1900).—This well-known open cluster, situated in the bright galactic clouds in Scutum Sobieski, is very rich in stars, several hundred being brighter than the fourteenth magnitude. Magnitudes and colors for a large number of stars in the cluster have been determined by Shapley.[2]

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[1] "Durchmusterung of Selected Areas," *Harvard Annals*, 101, 1918.

[2] *Mt. Wilson Contr.*, No. 126; *Astrophysical Journal*, 45, 164, 1917.

Table IX gives the results of the classification of spectra for eighty-five stars in the cluster. A great many stars which are sufficiently bright have their images superposed, thus making estimates of their spectral types impossible. For early types the classification is based on the strength of the K line relative to the hydrogen lines; for the late types the spectra have been compared with spectrograms of stars with known spectroscopic parallaxes, photographed with the same instrumental arrangement on the same kind of plates. The numbers are those of Stratonoff[1]; the photographic magnitudes are according to Shapley. The letter *d* after the spectral type means for an A-type star that the spectrum shows a marked energy-fall at λ 3907, thus indicating a relatively low luminosity ($M > 1.5$); for later types a *g* indicates giant characteristics, in this case strong "cyanogen" absorption in the region λ 4144–λ 4184.

TABLE VIII

LIST OF PLATES

Plate No.	Object	Date	Exposure-Time	Harvard No.
33	Sel. Area 63	1921 May 10	$1^h 43^m$	527
35	Sel. Area 63	May 11	2 55	527
46	Messier 11	Aug. 2	30	....
47	Messier 11	Aug. 2	2 0	....
48	Messier 11	Aug. 2	5 1	....
49	Sel. Area 64	Aug. 2	4 0	666
50	Messier 13	Aug. 3	1 30	....
51	Sel. Area 64	Aug. 3	4 2	509
65	Sel. Area 64	Sept. 4	3 30	1029
75	Sel. Area 64	Sept. 5	4 37	454

The giants of types K0–K2 in Table IX have nearly the same apparent magnitude. The estimate $M = 0$ for their absolute magnitude seems to be fair, judging from the very marked "cyanogen" absorption in their spectra. The mean photographic magnitude of the four stars Nos. 252, 423, 666, 674, is 13.32. With a mean color-index somewhat exceeding unity, we find the modulus $M - m$ to be nearly equal to -12.

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[1] *Publications de l'Observatoire de Tachkent*, No. 1, 1899.

The two brightest A stars in Table IX, Nos. 545 and 437, show sharp lines suggesting very high luminosity. With the value $M-m=-12$ their absolute magnitudes would be -2.8 and -0.5, respectively. The remaining A stars have fairly broad hydrogen lines. Between photographic magnitude 11.8 and 13.5 the energy-distribution around λ 3907 was found to be smooth, except for star No. 663; with the value of $M-m$ found above, those limits will correspond to $M=-0.2$ and $M=+1.5$. The faintest A stars in the tables show indications of a more abrupt fall of intensity at

TABLE IX

MESSIER 11

No.	Pg. Mag.	Spectral Type	No.	Pg. Mag.	Spectral Type	No.	Pg. Mag.	Spectral Type
193....	....	A0	365...	13.32	A0	571....	13.16	A0
197....	....	A2	366...	13.02	A0	579....	(Triple)	A0*d*
204....	13.38	A0	375...	12.54	A0	581....	11.85	A0
214....	13.48	A0	377...	13.64	A0	583....	13.84	A2*d*
221....	13.07	A0	381...	12.44	A0	590....	13.04	(G8*g*)?
224....	13.56	A0*d*	405...	13.18	A0	592....	14.22	A0*d*
227....	12.06	A0	407...	11.93	A0	597....	12.78	A2
232....	12.22	A0	423...	13.46	K2*g*	598....	13.02	G8*g*
236....	13.07	A0	425...	12.62	A2	599....	12.40	A0
237....	11.74	A0	426...	13.21	A0*d*?	601....	11.69	F?
245....	12.15	A0	437...	11.46	A0	616....	13.89	A2*d*
252....	13.08	K*g*	439...	13.64	A0	617....	13.25	A0
267....	13.26	A0	460...	12.97	A0	630....	13.54	A0
273....	13.00	A0	462...	12.79	A0	631....	12.46	A0
274....	14.32	A2*d*?	464...	13.94	A0	633....	13.37	A0
280....	13.01	G8*g*	468...	13.91	A2*d*?	635....	13.23	A0
285....	13.08	A0	486...	12.85	A0	647....	12.66	A0
288....	12.75	A3	487...	12.39	A2	662....	12.47	A0
297....	12.44	A0	490...	13.12	A0	663....	11.90	A5*d*
307....	14.20	A2*d*	491...	12.33	A0	666....	13.45	K0*g*
310....	13.04	A0*d*?	494...	12.32	A0	672....	12.50	A0
322....	13.46:	A0	511...	....	G8*g*	674....	13.28	K2*g*
325....	13.04	A2	519...	13.30	A0	704....	....	A0*d*?
331....	12.24	A0	525...	12.08	A0	715....	....	A0
336....	12.08	A0	529...	13.48	A0*d*?	762....	....	K2*g*
341....	12.42	A0	545...	9.3:	A5	....	....	....
345....	12.06	A0	548...	12.23	A0	....	....	....
356....	12.60	A0	552...	14.20	A0*d*?	....	....	....
362....	13.68	A0	568...	13.31	A0	....	....	....
364....	12.73	G8*g*	569...	13.26	A0	....	....	....

λ 3907. For Nos. 224, 307, 592, 616, where this is most noticeable, we find for M the values of 1.6, 2.3, 2.2, 2.1, respectively, which are in good agreement with the results for the A stars in the first part of this paper. The A5 star No. 663, photographic magnitude 11.90, is probably a relatively near star, superposed on the cluster. The spectra of the four faint A stars just mentioned are rather weak, and no definite conclusions can be drawn from those stars alone, but the indications are that the value $M-m=-12$ deserves a fair degree of confidence. Adopting this value we find for the parallax of the cluster $\pi=0\overset{''}{.}00025$, corresponding to a distance of 4000 parsec or 13,000 light-years. Shapley,[1] obtaining a value by an entirely different method, finds $\pi=0\overset{''}{.}00014$. In view of the uncertainties affecting the two determinations the difference cannot be considered as by any means excessive.

Globular clusters.—On Plate No. 50 three of the brightest stars in the globular cluster Messier 13 are sufficiently well separated from other stars to permit an examination of their spectra. They are the Scheiner Nos. 47, 63, 127. Shapley[2] gives the photographic magnitudes 13.45, 13.70, 13.58, and the color-indices 1.41, 1.16, 1.42. With the parallax $=0\overset{''}{.}000090$[3] the absolute magnitudes are -3.19, -2.69, -3.07, respectively. These stars seem to be of a spectral type not far from K0, but show a much smaller absorption in the region of the "cyanogen" band than the late-type stars in Messier 11. This agrees with the degree of luminosity found by Shapley, if the stars are pseudo-Cepheids; if they are ordinary giants we should estimate the absolute magnitude around $+1$. Low dispersion slit spectrograms of some stars in Messier 13 on a plate taken by Mr. Pease, and of stars in Messier 3 on a plate taken by Mr. Sanford, exposure-times 30^h and 20^h, respectively, seem rather to confirm the high degree of luminosity. The stars of earlier type on the first mentioned plate seem to have no appreciable change of intensity at λ 3907, and for the late-type stars on the latter plate, especially for the star von Zeipel No. 752, photographic magnitude 13.96, color-index 1.42, absolute magnitude according to

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[1] *Proceedings of the National Academy of Sciences*, **5**, 344, 1919.

[2] *Mt. Wilson Contr.*, No. 116.

[3] Shapley, *Mt. Wilson Contr.*, No. 152; *Astrophysical Journal*, **48**, 154, 1918.

Shapley −3.18, the depression between λ 4100 and λ 4227 seems to be centered around Hδ and λ 4172 rather than in the "cyanogen" band, observations in favor of a classification in the Pseudo-Cepheid type.

Selected Areas
No. 63, R.A. 19^h0^m, Decl.+30°0′, (1900)
No. 64, R.A. 19^h58^m, Decl.+30°0′.

TABLE X

SELECTED AREAS 63 AND 64

Harv. No.	Pg. Mag.	Spectral Type	Harv. No.	Pg. Mag.	Spectral Type	Harv. No.	Pg. Mag.	Spectral Type
Area 63			504....	14.5	A0	710...	12.87	A0
511....	13.20	K3*g*	524....	14.0	A2	719...	14.40	K2*g*
513....	12.61	G5*d*	535....	12.5	A0	726...	14.26	K2*g*
518....	9.61	A0	581....	12.5	B9	733...	11.70	A0
519....	10.50	A5	583....	12.0	A0	744...	13.06	F8
535....	12.43	G0*d*						
			603....	13.6	A2	748...	13.31	A0
539....	11.86	A5*d*	611....	13.8	F8	767...	12.10	G5*d*
546....	13.43	K2*g*	619....	14.6	A0	1005...	11.3	A0
			623....	14.2	A0	1007...	10.6	A0
Area 64			629....	13.7	A0	1017...	13.1	K0*g*
430....	14.6	A						
435....	13.2	K2*g*	646....	11.3	G8*g*	1020...	13.9	K2*g*
438....	13.8	F5*g*	661....	14.1	A0	1021...	13.6	B?
442....	13.2	G8*d*	667....	14.18	K5*g*	1026...	10.9	F0
443....	14.0	F8*d*	668....	13.0	F8*d*	1027...	13.8	G0*d*
			671....	14.22	K0*g*	1028...	13.3	A0
445....	14.0	K5*g*						
448....	13.9	G5*d*	677....	13.31	A0	1104...	11.8	A2*d*
476....	13.5	G5*d*	680....	13.86	K0*g*			
479....	14.1	A0	682....	13.64	G5*d*			
486....	12.7	G5*d*	702....	13.81	K2*g*			
			706....	10.52	A2			

The results of classification for fifty-two stars in Selected Areas 63 and 64 are given in Table X. The numbers are according to the Harvard Durchmusterung. The photographic magnitudes for the central fields are according to unpublished Mount Wilson measures; for stars more distant from the center of the area the Harvard magnitudes have been reduced to the Mount Wilson system, and are given to only one decimal.

These two areas are situated in very rich galactic regions. There are eleven late-type stars classified as dwarfs in Table X, which as a group may be considered to be the nearest stars in the region. Of the remaining more distant stars twenty-eight are fainter than the thirteenth magnitude, twelve of which are found to be giants of types K0–K5, a remarkably high percentage. These late-type giants range in photographic magnitudes from 13.1 to 14.4, with a mean of 13.8. With a mean color-index of 1.2 and an absolute visual magnitude of 0.5 they should be at the same distance as M 11, about 4000 parsecs. If M 11 is associated with the galactic clouds in Scutum, the inference would be that these giant stars together with some of the A stars are members of the distant galactic cloud in Cygnus.

I wish to express my deep feeling of obligation to Director Hale for the opportunity to undertake this investigation during a stay of more than a year at the Observatory, and also to several members of the staff, especially Mr. Adams and Mr. Seares, for much valuable information and advice. I also wish to express my gratitude to Director Campbell for a month's stay at the Lick Observatory, and to Mr. Wright for his kind help and instruction in the work with the Crossley reflector.

MOUNT WILSON OBSERVATORY
October 1921

THE THEORY OF IONIZATION AND THE SUN-SPOT SPECTRUM[1]

By HENRY NORRIS RUSSELL[2]

ABSTRACT

Saha's theory of temperature ionization, assuming the ionization to be due to temperature alone, leads to the relation: $Px^2/(1-x^2)=K$; that is, the ratio of the fraction of atoms ionized (x) to the fraction un-ionized ($1-x$), times the partial pressure of the free electrons [$Px/(1+x)$], is equal to K, a function only of the ionization voltage I and of the absolute temperature T, namely: $\log_{10} K=-5036\,I/T+2.5\log_{10} T-6.5$. The theory is here *extended to mixtures of elements* by putting the partial pressure of the electrons equal to $P\bar{x}/(1+\bar{x})$ where $\bar{x}$ is the fraction of ionization for all the atoms present, and P, as before, is the total pressure. Hence, $Px_1\,\bar{x}/(1-x_1)(1+{}^{-})=K_1$; $Px_2\,\bar{x}/(1-x_2)(1+\bar{x})=K_2$; etc.; and the ratio of $x_1/(1-x_1)$ to $x_2/(1-x_2)$ is merely K_1/K_2. Therefore the degree of ionization is always higher for an element of easy than for one of difficult ionization, and the relation between the two depends only on (I_2-I_1) and on T, and not on the pressure or on the relative concentrations. Saha's *formulae for second-stage ionisation* are corrected. Only two successive stages of ionization can be present in any considerable proportions at the same time. Strong *evidence in favor of the theory* is afforded by a comparison of the sun-spot and the solar spectrum, since the predictions of the theory with reference to the relative intensity in the hotter and the cooler spectrum, of lines associated with ionized and un-ionized atoms, are found to be in general agreement with the facts. However, discrepancies such as the presence of too large a fraction of ionized Ba atoms, suggest the need of some modification of the theory; and it is evidently incomplete, since it neglects the effects of radiation on ionization.

Relative intensity in solar and sun-spot spectra of series lines.—The lines of the *alkali metals* (all of which are due to the neutral atoms) are *greatly strengthened* in the spot spectrum. Na ($I=5.11$ volts) exhibits its principal and subordinate series, K (4.32) the principal series alone, and Li (5.37) and Rb (4.16) the leading pair of the principal series, and only in the spot spectrum. The corresponding lines of Cs are out of reach in the infra-red.

Among the *alkaline earths* the lines of the *neutral atom* for Ca (6.08 volts) are strengthened in the spots to a degree which is greatest for the principal series and least for the combination series. The corresponding lines of Sr (5.67) are faint in the sun and much strengthened in spots. For Ba (5.12 volts) they are absent both in sun and spots. The *enhanced lines* arising from the ionized atoms (Ca+, Sr+, Ba+) are strong in both spectra.

For Mg (7.65 volts) the lines of the *neutral atom* are little affected, except that the fundamental line $1S-2p_2$ is stronger, while for Zn (9.4 volts) the arc lines are much weaker in the spot spectrum.

These results are all in agreement with Saha's theory. The lines of Li and Ba, however, are much weaker than the theory indicates.

Relative intensity of arc and enhanced lines.—It is pointed out that enhanced lines of easily ionized elements can be produced with less excitation than arc lines of difficultly ionized elements. This fact explains many apparent anomalies in astrophysical spectra.

Ionization potential of the elements as a function of atomic number.—Since from the relative behavior in sun, spot, and furnace spectra, of the lines of *Ca*, *Sc*, *Ti*, *V*, *Cr*,

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[1] *Contributions from the Mount Wilson Observatory*, No. 225.

[2] Research Associate of the Mount Wilson Observatory.

Mn, Fe, Co, Ni, Cu, and Zn, the elements as arranged in the order of their atomic numbers, are progressively more difficult to excite spectroscopically, the ionization potential probably increases regularly from 6 volts for Ca to 9.4 volts for Zn, and this result combined with other data suggests that, *in general,* the ionization potential is a periodic function of the atomic number.

Identification of certain solar lines.—Lines of Rb have been detected in the sun for the first time. Also several faint Na lines have been identified and Datta's identifications of Na and K lines have been partly confirmed, partly not.

Tables of the series lines of Li, Na, K, Rb, Cs, Ca, Sr, Ba, Mg, and Zn are given with intensities in sun and spot, and wave-lengths in international units to 0.01 A.

Dr. Megh Nad Saha, in an important series of papers,[1] has shown the great importance of the modern thermodynamic theory of ionization in astrophysical and spectroscopic researches. In particular, he makes certain predictions about the behavior of the lines of the alkali metals in the spectra of sun-spots. The present communication deals with certain points connected with the theory, especially when many kinds of atoms are present at the same time, and with the observational verification of the predictions, which has been complete.

I. THE THEORY OF IONIZATION WHEN ATOMS OF SEVERAL KINDS ARE PRESENT

The equation of the "reaction isobar" as given by Saha[2] in the case of the ionization of a gas consisting of atoms of but a single kind is

$$\log \frac{x^2}{1-x^2}P=\log K=-\frac{U}{4.571\,T}+2.5\log T-6.5 \qquad (1)$$

where the logarithms are to the base 10, P is the pressure of the gas, x the fraction of all the atoms which are ionized, T the absolute temperature centigrade, and U the "heat of ionization" in calories per gram-molecule. For the gas in question, the latter is proportional to the ionization potential, I, being 23,020 calories if I is one volt, so that we may also write

$$\frac{x^2}{1-x^2}=\frac{K}{P}, \qquad \log K=-\frac{5036\,I}{T}+2.5\log T-6.5. \qquad (2)$$

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[1] "Ionization in the Solar Atmosphere," *Philosophical Magazine*, 40, 472, 1920; "Elements in the Sun," *ibid.*, 40, 809, 1920; "On the Temperature Radiation of Gases," *ibid.*, 41, 267, 1921; "On a Physical Theory of Stellar Spectra," *Proceedings of the Royal Society*, A, 99, 135, 1921.

[2] *Philosophical Magazine*, 40, 479, 1920.

If atoms of several different kinds, all capable of ionization, are present, the situation is somewhat more complicated. To use equation (1), introducing for P the value of the partial pressure of the vapor of each element separately, is inadmissible, since one of the products of the reaction—free electrons—is produced by all the ionizations. But if p' is the partial pressure of the ionized atoms of any sort, p that of the corresponding non-ionized atoms, and p'' that of the free electrons, we will always have

$$\frac{p'p''}{p}=K \tag{3}$$

when K is given by (2).

If now atoms of several kinds are present in the relative numbers $a_1, a_2, a_3 \ldots$ and the fractions $x_1, x_2, x_3 \ldots$ of these are ionized, the whole number of atoms and free electrons present will be $a_1(1+x_1)+a_2(1+x_2) \ldots$ and the partial pressures for element 1 will be

$$p_1'=\frac{a_1x_1}{\Sigma a+\Sigma ax}P \qquad p_1=\frac{a_1(1-x_1)}{\Sigma a+\Sigma ax}P$$

(P being as before the whole pressure of the gas), while for the free electrons we will have

$$p''=\frac{\Sigma ax}{\Sigma a+\Sigma ax}P=\frac{\bar{x}}{1+\bar{x}}P$$

where

$$\bar{x}=\frac{\Sigma ax}{\Sigma a}. \tag{4}$$

$\bar{x}$ is therefore the fraction of all the atoms present which are ionized, and may be regarded as a weighted mean of the values of x for the individual elements.

Equation (3) then becomes for the various elements

$$\frac{x_1}{1-x_1}\frac{\bar{x}}{1+\bar{x}}=\frac{K_1}{P}, \qquad \frac{x_2}{1-x_2}\frac{\bar{x}}{1+\bar{x}}=\frac{K_2}{P}, \ldots \tag{5}$$

where K_1, K_2 are given by (2) with the ionization potentials for the elements concerned.

We then have

$$\frac{x_1}{1-x_1}=\frac{K_1}{K_2}\frac{x_2}{1-x_2}, \qquad \log\frac{K_1}{K_2}=5036\,\frac{I_2-I_1}{T} \tag{6}$$

or, expressed verbally: *The ratios of the number of ionized atoms to that of non-ionized atoms for any two elements in a gaseous mixture bear a fixed proportion to one another, which depends only on the temperature, and is independent of the pressure, the relative abundance of the two elements, or the presence of other elements.*

These latter conditions affect the amount of ionization, without modifying this proportion. The element of lower ionization potential is always the more highly ionized.

We may write (5) in the form

$$\frac{x_1^2}{1-x_1^2}=\frac{K_1}{P_1},$$

where

$$P_1=P\frac{\bar{x}}{x_1}\frac{1+x_1}{1+\bar{x}}. \tag{7}$$

If $x_1>\bar{x}$, that is, if the element is more easily ionized than the average, P_1 will be less than P, and the percentage of ionization will be greater than it would be if this element alone were present, with the same total pressure. For elements ionized with more difficulty than the average, the reverse will be the case.

The ionization will in all cases be less than if this element were present alone with its actual partial pressure, for p'', in equation (3), will always be greater than the partial pressure p_1' of the electrons arising from the dissociation of this one component of the mixture.

When ionization is just beginning all the x's will be small, and we will have approximately

$$x_2=\frac{K_2}{K_1}x_1,\ \text{etc.} \qquad \bar{x}=\frac{x_1\Sigma aK}{K_1\Sigma a}$$

$$P_1=P\frac{\bar{x}}{x_1}=P\frac{a_1}{\Sigma a}\left(1+\frac{K_2a_2}{K_1a_1}+\frac{K_3a_3}{K_1a_1}\ \ldots\ \right).$$

The factor outside the parentheses is the partial pressure of constituent 1. It follows that at the beginning of ionization the constituent of lowest ionization potential behaves very nearly as if it alone were present at its actual partial pressure. For constituents more difficult of ionization the initial ionization is less, and may be very much smaller.

When the ionization of any element is nearly complete, we may set $x_1=1$ in (7) and find

$$P_1=P\frac{2\bar{x}}{1+\bar{x}}.$$

The effective pressure for this element is therefore nearly equal to the total pressure (unless a large proportion of atoms of more difficult ionization is present).

For an element of easy ionization the range of temperature or pressure within which both ionized and uncharged atoms will be present in sensible proportions is therefore extended by the presence of other constituents in the mixture.

For an element of difficult ionization, x_1 will be much less than $\bar{x}$ and P_1 greater than P, when ionization is beginning. The range within which both phases are present will in this case be reduced.

If we set

$$\frac{\bar{x}^2}{1-\bar{x}^2}=\frac{\bar{K}}{P}$$

we find easily

$$\bar{K}\frac{1-\bar{x}}{\bar{x}}=K_1\frac{1-x_1}{x_1}=K_2\frac{1-x_2}{x_2}\;\ldots\ldots=\frac{\Sigma aK(1-x)}{\Sigma ax}$$

$$\bar{K}=\frac{\Sigma aK(1-x)}{\Sigma a(1-x)}$$

so that $\bar{K}$ is the mean of the individual K's, weighted in proportion to the number of non-ionized atoms remaining. At low temperatures, the atoms of easy ionization, which have the largest values of K, will contribute most to this mean; but at high temperatures these will be ionized, and the atoms of difficult ionization will contribute most. The effective mean ionization potential, corresponding to the value of $\bar{K}$, will therefore increase with the temperature.

Successive ionization.—Consider now atoms of the same kind, which are susceptible of the loss of two successive electrons. Let the ionization potential for the removal of the first be I and the additional potential for the second be I'. If x is the proportion of singly-ionized and y of doubly-ionized atoms, that of free electrons will be $x+2y$, and we will have[1]

$$\left.\begin{aligned}\frac{x(x+2y)}{(1-x-y)(1+x+2y)}&=\frac{K}{P}\\ \frac{y(x+2y)}{x(1+x+2y)}&=\frac{K'}{P}\end{aligned}\right\}. \tag{8}$$

The first of these equations may be written

$$\frac{(x+y)^2}{1-(x+y)^2}=\frac{K}{P}\,\frac{(x^2+2xy+y^2)(1+x+2y)}{(x^2+2xy)(1+x+y)}>\frac{K}{P}$$

also

$$\frac{x^2}{1-x^2}=\frac{K}{P}\left(1-\frac{y(2-x+x^2+2xy)}{(1-x^2)(x+2y)}\right)<\frac{K}{P}$$

while the second may be written

$$\frac{y^2}{1-y^2}=\frac{K'}{P}\,\frac{x}{1-y}\,\frac{2y^2+xy+y}{2y^2+xy+2y+x}<\frac{K'}{P}.$$

It follows that the numbers of both neutral and singly-ionized atoms are less than they would be if the second ionization did not occur—the draft upon the singly-ionized atoms being partly made up by fresh ionization of neutral ones; but the number of doubly-ionized atoms is less than it would be if the second ionization was the only one which occurred.

Dividing the second of the equations (8) by the first we find

$$\frac{y(1-x-y)}{x^2}=\frac{K'}{K}. \tag{9}$$

Now

$$\log\frac{K'}{K}=-5036\frac{I'-I}{T}$$

and in all cases which have so far been investigated $I'-I$ is positive, and equal to at least five volts.

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[1] These equations differ from Saha's (*Proceedings of the Royal Society*, A, **99**, 143, 1921) which contain x^2 and y^2 in the numerators; but the latter formula would appear to involve the inadvertent assumption that an atom can combine only with an electron previously liberated by another atom which happens to be in the same state of ionization.

Hence K'/K is usually a small quantity, and it is impossible for y and $1-x-y$ to be considerable at the same time; that is, there will not simultaneously be any noticeable proportion of atoms in all three states of ionization. When the temperature is high K'/K increases; but in this case x is small and the number of neutral atoms very small.

At low temperatures we may set $y=0$, when the ordinary equation for single ionization is obtained; at high temperatures we may set $x+y=1$ and have

$$\frac{y^2+y}{2-y^2-y}=\frac{K'}{P}.$$

One or the other of these formulae will always give a good first approximation, provided that a third electron is not lost.

In a mixed gas, containing other constituents, we will have

$$\frac{x\bar{x}}{(1-x-y)(1+\bar{x})}=\frac{K}{P};\quad \frac{y\bar{x}}{x(1+\bar{x})}=\frac{K'}{P};\quad \bar{x}=\frac{\Sigma a(x+2y)}{\Sigma a}. \tag{10}$$

Since the potential for the second ionization is always high, the presence of other elements will in general diminish its amount; but equation (9) will hold in all cases.

Instead of equation (6) we will have

$$\frac{x}{1-x-y}=\frac{K}{K_2}\frac{x_2}{1-x_2}\qquad \frac{y}{x}=\frac{K'}{K_2}\frac{x_2}{1-x_2} \tag{11}$$

where x_2 is the percentage of ionization for any singly ionized element.

2. THE ALKALIES AND ALKALINE EARTHS IN THE SUN-SPOT SPECTRUM

Saha has shown[1] that sodium should be considerably ionized in the reversing layer, potassium almost completely, and rubidium and caesium entirely. As the familiar lines of these elements are absorbed by the neutral atoms only, the faintness of the lines of potassium in the solar spectrum and the absence of those of rubidium and caesium is immediately explained. At the lower temperatures which prevail in the spots he predicts that the lines of potassium should be strengthened, and that those of rubidium should show faintly.

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[1] *Philosophical Magazine*, **42**, 812–815, 1920.

Observations upon the behavior of sodium in sun-spots have been published by Adams[1] and the presence of lithium has been determined by King.[2] The study of the other alkali metals, which have their lines principally in the red, has been made very easy by the publication of extensive and accurate tables of wave-lengths in the red and infra-red, both in the solar spectrum[3] and in the arc,[4] and by the existence at Mount Wilson of an excellent series of plates of the spot spectrum, extending to λ 8200, taken by Mr. Brackett with the 150-foot tower telescope and 75-foot spectrograph, in the first order, with nicol prism and compound quarter-wave plate. All the available lines of the alkali metals were examined on these plates (or on the photographic map of the spot spectrum for the wave-lengths less than 6600 A) and some additions to the data for sodium and lithium were made.

The results, including those of the earlier observers, are given in Table I. In this table the first column gives the element and series designation of the line (after Sommerfeld). The leading pair of the principal series is denoted by 1s–2p, the next pair by 1s–3p, and so on. The corresponding pairs for the diffuse series are 2p–3d, 2p–4d, and for the sharp series 2p–3s, etc. Lines lying too far in the infra-red to be observed, or too far in the ultra-violet to be observable in spots, are in general omitted, as are also the fainter members of a series when the stronger earlier members are invisible.

The second column gives the wave-lengths found in the laboratory by Datta or by Meggers (or, in one or two cases, by others); and the third, the wave-length of the corresponding solar line derived from Rowland's tables, from Meggers, or from measurements made at Mount Wilson. All wave-lengths are given in

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[1] *Mt. Wilson Contr.*, No. 40; *Astrophysical Journal*, **30**, 108, 1909.

[2] *Mt. Wilson Contr.*, No. 122; *Astrophysical Journal*, **44**, 169, 1916.

[3] W. F. Meggers, "Solar and Terrestrial Absorption in the Sun's Spectrum from 6500 A to 9000 A," *Publications of the Allegheny Observatory*, **6**, 12–44, 1919.

[4] W. F. Meggers, "Wave-Length Measurements in Spectra from 5600 A to 9600 A," *Bulletin of the Bureau of Standards*, **14**, 371–397, 1916; S. Datta, "The Vacuum Arc Spectra of Sodium and Potassium," *Proceedings of the Royal Society*, A, **99**, 69–78, 1921.

TABLE I

Lines of the Alkali Metals

Element	Wave-Lengths		Intensity		Authority	Notes
	Arc	Sun	Sun	Spot		
Lithium						
1s–2p	6707.82	07.83	abs	5	King	1
2p–2s	8126.52	....	abs	abs	Russell	
2p–3d	6103.53	....	abs	abs	King	
Sodium						
1s–2p	5895.94	95.94	20	60	Adams	
	5889.97	89.97	30	90	Adams	
2p–3d	8194.82	94.84	2	7	Russell	
	8183.30	83.27	3	10	Russell	
2p–4d	5688.22	88.22	6	12	Adams	
	5682.67	82.65	5	12	Adams	
2p–5d	4982.86	82.82	2	4	Russell	2
	4978.61	78.56	0	1	Russell	3
2p–6d	4668.60	68.57	1*N*	3	Russell	4
	4664.86	....	....	....	Russell	5
2p–7d	4497.72	97.73	abs	0	Wilson	6
	4494.27	....	abs	abs	Russell	7
2p–4s	6160.72	60.75	3	8	Adams	
	6154.21	54.23	2	9	Adams	
2p–5s	5153.85	....	abs	abs	Russell	8
	5149.27	....	abs	abs	Russell	9
2p–6s	4751.89	51.83	00	2	Adams	
	4748.02	47.98	000	0	Adams	
Potassium						
1s–2p	7699.00	99.01	5	13	Russell	
	7664.92	64.92	6	15	Russell	10
1s–3p	4047.20	47.19	00*Nd*	0	Russell	11
	4044.14	44.14	0	2	Russell	
2p–4d	6965.0	....	abs	abs	Russell	12
	6936.0	....	abs	abs	Russell	
2p–5d	5832.31	....	abs	abs	Russell	
	5812.71	....	abs	abs	Russell	
2p–4s	6939.50	....	abs	abs	Russell	13
	6911.80	....	abs	abs	Russell	
2p–5s	5802.16	....	abs	abs	Russell	
	5782.77	....	abs	abs	Russell	

TABLE I—*Continued*

Element	Wave-Lengths In		Intensity In		Authority	Notes
	Arc	Sun	Sun	Spot		
Rubidium						
1s–2p.........	7947.64	47.63	abs	0	Russell	14
	7800.29	00.29	abs	1	Russell	
1s–3p.........	4215.57	....	abs	abs	Russell	
	4199.85	....	abs	abs	Russell	
2p–4d.........	7757.80	....	abs	abs	Russell	
	7619.12	....	abs	abs	Russell	
Caesium						
1s–2p.........	8943.46	....	abs	Unknown	....	15
	8521.12	....	abs	Unknown	....	
1s–3p.........	4593.22	....	abs	abs	Russell	
	4555.31	....	abs	abs	Russell	
	8079.4	....	abs	abs	Russell	16
	8016.2	....	abs	abs	Russell	

NOTES TO TABLE I

1. Much widened in the spot, and showing strong Zeeman effect.

2. Rowland's line 4982.994, for which he gives no origin. Identified by Datta.

3. Rowland 4978.732. Shows, like the last, a conspicuous Zeeman effect. Clearly separated from the iron line 4978.785 which is erroneously identified by Datta as the sodium line.

4. Rowland 4668.749. Shows Zeeman effect. Identified by Datta.

5. Masked by the chromium line 4664.965 (Rowland).

6. A faint line, visible only in the spot, and showing conspicuous Zeeman effect. Measured by Wilson (unpublished) 4497.900 on Rowland's scale. Datta identifies this with Rowland 4497.842, a titanium line which is clearly separated.

7. Datta identifies this with Rowland 4494.356 (00) which is not strengthened in the spot, and is probably not the sodium line. No trace of the latter could be seen. The character of the foregoing lines, and their strong intensification in the spots, support their identification as sodium lines. It is very difficult to get these lines sharp in the laboratory, and Datta's measures, though much superior to earlier ones, demanded long exposures.

8. Datta identifies as Rowland 5153.848 (000). This is a band line showing no Zeeman effect.

9. Datta identifies as 5149.267 (000) Rowland C. This is very little strengthened in the spot and shows no Zeeman effect. No trace of the sodium lines could be found in this vicinity.

10. Practically coincident with a strong atmospheric line in the tail of the A band, but identified by Meggers by means of the sun's rotation. The intensification of this line in the spot spectrum and the Zeeman effect are conspicuous in spite of this blending, which has been allowed for in the estimates of intensity.

11. Identification by Rowland and Datta confirmed.

12. Wave-lengths from Saunders. These lines are abnormally faint in the arc spectrum. Datta identifies as Rowland 6965.320 (000 N) which shows no Zeeman effect and 6926.290 (0000) which is invisible both in sun and spot on the Mount Wilson plates.

13. Identified by Datta as 5832.490 (000) and 5812.942 (0000). These lines are too faint to be visible on the Mount Wilson plates, either in sun or spot.

14 Typical spot lines, diffuse and showing strong Zeeman effect, and greatly resembling the lithium line at λ 6708. Measured by the writer, the first on one plate, the second on two. Probable error of measurement about ±0.01 A. One of the Zeeman components of 7947.63 is clearly separated from the atmospheric line at 7947.766 (Meggers). The other is blended with it.

15. These lines, which are the first pair of the principal series, lie beyond the limits of the region so far photographed at Mount Wilson, and attempts to secure plates showing the spot spectrum in this region have not yet been successful. They are not present in the solar spectrum (Meggers). The absence of the second pair of this series might have been anticipated.

16. This relatively narrow pair is strong in the arc, but does not belong to the known series. There are no strong lines of the subordinate series of caesium in favorable positions for investigation.

International units and Rowland's values have been reduced to this standard with the aid of a correction-curve which was kindly furnished by St. John.

The fourth and fifth columns give the intensity of the lines in sun and spot. The values in the former are taken from Rowland or Meggers; the latter, for the lines previously observed, from Adams or King. For lines newly observed, estimates have been made on the original plates, as nearly as possible upon the same scale as that used by Rowland or Meggers. Cases in which a line is certainly invisible are denoted by "abs." The sixth column gives the authority for the data concerning the spot spectrum (including measures of wave-length when the line does not appear in the spectrum of the photosphere), and the last provides references to the notes which follow.

These results may be summarized as follows:

Sodium is represented in the sun by the principal, diffuse, and sharp series, and all its lines are much strengthened in the spot spectrum. Some of Datta's identifications of faint lines have been confirmed and others modified. The absence of the pair λλ 5153, 5149 in both sun and spot is remarkable, since the following and fainter pair of the series appears to be present.

Potassium is represented by the principal series only. Its lines are greatly strengthened in the spots. Datta's identifications of faint solar lines as belonging to the subordinate series of this element are not confirmed.

Lithium shows in the spot spectrum alone, and only through the leading line of the principal series.

Rubidium, previously undetected in the sun, is definitely proved to be present by the appearance of both members of the leading pair of the principal series in the spot spectrum.

The corresponding lines of caesium lie far in the infra-red, and satisfactory photographs of the spot spectrum have not yet been obtained in this region.

The alkaline earths (for which the ionization potentials are likewise known) are of special interest, because the fundamental members 1s–2p, of the series of enhanced lines, absorbed by the ionized atoms, lie within the accessibie portion of the solar spectrum

TABLE II

LINES OF ALKALINE EARTHS IN SUN-SPOTS

ELEMENTS	WAVE-LENGTH	INTENSITY IN		NOTES
		Sun	Spot	
Mg				
1S–2P	2852.13	....	....	Out of reach
1S–2P_2	4571.11	5	7	1
2P–4D	5528.47	8	8	1
2P–5D	4703.07	10	10	1
2P–6D	4351.94	5	4	1
2P–4S	5711.13	6	6	1
2p–3s	5183.62	30	30	1
	5172.68	20	22	1
	5167.33	15	17	1
2p–4d	3838.28	25	....	
	3832.31	15	....	7
	3829.36	10	....	
Mg+				
1s–2p	2802.70	....	....	Out of reach
	2795.52	....	....	
Ca				
1S–2P	4226.73	20	25	2
1S–2p_2	6572.78	1	10	3
2P–3D	7326.10	2	5	4
2P–4D	5188.85	3	4	4
2P–4S	5512.98	4	5	4
2P–5S	4847.29	0	1	4
3D–4F	4878.13	3	3	4
3D–2P	6717.19	5	8	4
2p–4d	4556.61	2	4	2
	4555.88	3	5	2
	4554.77	5	7	2
	4535.67	4	6	2
	4534.95	5	7	2
	4525.43	4	6	2
2p–2s	6162.18	15	25	2
	6122.22	10	20	2
	6102.72	9	20	2
3d–4f	4585.92 }	4	6	4
	4585.87 }	....	....	
	4581.41	4	6	4
	4578.57	3	6	4
Ca+				
1s–2p	3968.08	700	700	3
	3933.68	1000	1000	3
2p–3d	8662.42	15	....	
	8542.48	16	....	
	8498.32	12	....	
2p–2s	3736.92	5	....	7
	3706.03	6	....	7
Sr				
1S–2P	4607.35	1	3	4
1S–2p_2	6892.36	abs	abs	4
2p–3s	7070.7	abs	abs	4
	6878.8	abs	abs	4
	6791.4	abs	abs	4

TABLE II—*Continued*

Elements	Wave-Length	Intensity in		Notes
		Sun	Spot	
Sr				
2p–4d	4971.68	abs	abs	4
	4967.94	....	....	5
	4962.28	000	00	4
	4876.06	abs	abs	4
	4872.49	abs	000	4
	4852.06	abs	abs	4
Sr+				
1s–2p	4215.50	8	8	2
	4077.73	5	5	2
2p–4d	10915.0	....	....	
	10328.3	....	....	Out of reach
	10038.7	....	....	Out of reach
2p–2s	4305.44	(3)	(5)	5
	4161.79	1	00	2
Ba				
1S–2P	5535.53	abs	abs	6
1S–2p_2	7911.36	abs	abs	4
Ba+				
1s–2p	4934.07	7	7	4
	4554.04	8	10	4
2p–4d	4166.01	0	0	4
	4130.65	2	2	4
	3891.83	....	....	5
2p–2s	4899.97	(2)	(2)	5
	4524.95	0	0	4
Zn				
1S–2P	2139.33	....	....	Out of reach
1S–2p_2	3075.90	1	....	7, 8
2P–3D	6362.35	1	0	9
2p–2s	4810.54	3	0	4
	4722.16	3	0	4
	4680.14	1	00	4
2p–4d	3346.02	0	....	7
	3345.58	1	....	7
	3345.12	2	....	7
	3303.37	....	....	5
	3303.59	1	....	7
	3282.33	....	....	5

NOTES

1. Adams, *Mt. Wilson Contr.*, No. 40, p. 20; *Astrophysical Journal*, 30, 105, 1909.
2. Adams, *Mt. Wilson Contr.*, No. 22; *Astrophysical Journal*, 27, 45–55, 1908.
3. Adams, *Mt. Wilson Contr.*, No. 40, p. 7; *Astrophysical Journal*, 30, 93, 1909.
4. From examination of the map of the sun-spot spectrum or of original plates for wave-lengths greater than 6600. Many of these lines have previously been studied at Mount Wilson, but they have been re-examined because the later plates are probably better.
5. Masked by other lines.
6. This line (according to the accurate measures of Schultz (*Zeit. für Wiss. Phot.*, 11, 209, 1912) lies at 5535.755 on Rowland's scale, and falls in a very confused region between the iron line 5535.644 (intensity 2) and the line 5535.778 (intensity 0), for which Rowland gives no origin. Between these two there is a faint line which is confined to the spot, in which it is of intensity 0 and shows Zeeman effect. It was measured and found to be at 5535.703. There is also a faint band line close by showing no Zeeman effect, but nothing which can be identified as the barium line. It is either absent, or masked by 5535.778.
7. Too far to the violet to show changes in the spot spectrum.
8. Rowland's line 3076.002, for which he gives no origin. Kayser and Runge find 3075.99 in the arc; Exner and Haschek, 3076.02.
9. But little weakened in the photographic map. Weakening confirmed on an original negative. Zeeman effect stronger over penumbra of spot than over umbra. May be blended with a band line. These are numerous in this region.

(except for magnesium) which is not known to be the case for any other elements. They are also noteworthy for possessing two radiation potentials, corresponding to the shift of an electron from its normal position in the neutral atom to the position corresponding to the limits of the subordinate series of single lines and of triplets.

These lines afford the most delicate tests for the presence of the ionized or neutral atoms of these elements. Their behavior in spots is indicated in Table II. Various other lines of these elements and the observable lines of zinc are included.

This table exhibits the well-known fact that the strongest lines of the alkaline earths (except magnesium) are enhanced lines.

The series of calcium and magnesium, both of triplets and of single lines, are conspicuous in the solar spectrum, and only some of the more prominent lines are given in the table. The calcium lines are considerably strengthened in the spot spectrum; those of magnesium are not, with one exception.

In the case of strontium, only the strongest lines of the neutral atom appear in the sun; these are very faint, but are intensified in spots, while not one of the strong low-temperature lines of barium appears either in the sun or the spots. This has been confirmed by examining a number of those which did not belong to known series and are therefore not given in the table.

3. APPLICATION OF THE THEORY OF IONIZATION TO THESE DATA

The ionization potentials of a number of elements of astrophysical importance are as follows:

	Volts		Volts		Volts
Caesium	3.81	Barium	5.12	Ba+	9.86
Rubidium	4.16	Strontium	5.67	Sr+	10.70
Potassium	4.32	Calcium	6.08	Ca+	11.86
Sodium	5.11	Magnesium	7.65	Mg+	15.02
Lithium	5.37	Zinc	9.4	Zn+	18.20
Hydrogen	13.54	Helium	25.2	He+	53.4

The last column gives the potentials corresponding to the additional energy required to remove a second electron after the first is already gone.

We may now proceed to calculate the percentages of the atoms of various elements which are neutral, singly-ionized, and doubly-ionized, in a gaseous mixture at the temperature of the reversing layer and of a sun-spot—which we will take in round numbers as 6000° and 4000° respectively—and at various pressures.

Since the pressure in the solar atmosphere is unknown within very wide limits (except that it is probably small) we may save trouble by computing for assigned values of the "effective pressure" P_1, for sodium, using equations (2) and (7). The difference between the effective pressure and the true pressure will be much less than the range of uncertainty of the latter. We may then find the percentages of ionization of other elements by (5) or (11).

The results are given in Table III. Neutral atoms are denoted as usual by Ca; singly-ionized, by Ca+; and doubly-ionized, by Ca++ (taking calcium as an example). The percentage of atoms in the highest stage of ionization for each element is not given, except for hydrogen, since it can easily be found by subtracting the sum of the others from 100, and also because atoms of the elements given in the table, when in these states, absorb no known lines in the visible spectrum.

It should be remembered that the equations from which these tables have been calculated are based upon certain assumptions, as yet unverified, namely, that the chemical constant for the electron may be derived by an extrapolation of the Sackur-Tetrode relation (which gives this constant as a function of the atomic weight of an element) to the electron, considered as having an atomic weight of 0.00055, and also that the chemical constant is the same for the neutral and ionized atoms of any element.[1] An error in the former assumption could be corrected by multiplying all the tabular pressures by a constant factor; one in the second, by multiplying the pressure for each element by an appropriate factor before entering the table.

What is more important, the tabular numbers depend on the assumption that temperature alone is effective in producing ionization. This is certainly not the case in the solar atmosphere, where the absorption of radiation, though it does not completely ionize the absorbing atoms, must undoubtedly get them into a state in which ionization by some other stimulus becomes easier.[1] This should operate to increase the ionization in the reversing layer to a considerable degree.

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[1] See Saha, *Philosophical Magazine*, 40, 479, 1920.

TABLE III

PERCENTAGES OF NEUTRAL AND IONIZED ATOMS

ELEMENT	*P* IN ATMOSPHERES						
	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
	Reversing Layer—Temperature 6000° C.						
Cs......	22.6	6.1	0.87	0.09	0.01	0.001	0.0001
Rb......	37.2	12.3	1.7	0.18	0.02	0.002	0.0002
K........	44.7	14.8	2.2	0.19	0.02	0.002	0.0002
Na......	79.3	44.4	9.6	1.1	0.11	0.011	0.0011
Li.......	86.1	57.0	15.0	1.9	0.19	0.019	0.002
Ba......	79.1	45.0	9.7	1.1	0.10	0.005	0.0001
Ba+....	10.9	55.0	90.2	98.0	91.3	48.3	9.6
Sr.......	91.7	70.2	23.8	3.2	0.3	0.03	0.001
Sr+.....	8.3	29.8	76.2	96.4	97.9	84.6	35.8
Ca......	96.1	84.0	41.0	6.8	0.7	0.07	0.007
Ca+....	3.9	16.0	59.0	93.2	99.1	98.0	83.6
Mg......	99.8	99.1	93.5	60.4	13.2	1.5	0.15
Mg+....	0.2	0.9	6.5	29.6	86.8	98.5	99.8
Zn......	99.99	99.96	99.7	96.9	75.6	23.7	3.1
Zn+....	0.01	0.04	0.3	3.1	24.4	76.3	96.9
H+.....	2×10^{-6}	1×10^{-5}	8×10^{-5}	8×10^{-4}	0.008	0.08	0.76
	Sun-Spot—Temperature 4000° C.						
Cs......	54.9	27.0	9.3	2.2	0.30	0.032	0.003
Rb......	77.0	50.0	21.9	5.7	0.83	0.089	0.009
K........	84.2	61.5	30.9	8.8	1.3	0.14	0.014
Na......	98.1	94.1	81.5	48.9	11.6	1.37	0.14
Li.......	99.2	97.1	90.3	66.9	21.8	2.95	0.29
Ba......	98.2	94.1	81.8	49.5	11.8	1.42	0.14
Ba+....	1.8	5.9	18.2	50.5	88.2	98.58	99.86
Sr.......	99.63	98.76	95.8	82.9	40.0	6.6	0.72
Sr+.....	0.37	1.24	4.2	17.1	60.0	93.4	99.28
Ca......	99.89	99.62	98.7	94.1	68.5	18.7	2.3
Ca+....	0.11	0.38	1.3	5.9	31.5	71.3	97.7
Mg.....	99.999	99.996	99.975	99.93	99.52	95.6	69.0
Mg+....	0.001	0.004	0.025	0.07	0.48	4.4	31.0
Zn......	100.00	100.00	100.00	100.00	99.995	99.95	99.51
Zn+....	1.3×10^{-5}	4.3×10^{-5}	1.6×10^{-4}	7.2×10^{-4}	0.005	0.05	0.49
H.+....	4.7×10^{-11}	1.6×10^{-10}	5.7×10^{-10}	2.6×10^{-9}	1.9×10^{-8}	1.8×10^{-7}	1.8×10^{-6}

In applying these results to the interpretation of the spot spectrum, it is probably safe to work on the assumption that the more abundant elements in the solar atmosphere are present in amounts much greater than is necessary to produce strong lines and that, for the principal lines of these elements (at least) the absorption is substantially "saturated" so that a considerable change in the amount of element would produce little change in these lines, except in the strength of the wings. It is also important to remember that, according to modern theory, the atoms of a given element, in their normal state, are ready to absorb certain fundamental lines (those of the principal series, denoted by 1s–2p, etc., above) while, in order that they may absorb other lines (such as those of the subordinate series) they must first have absorbed energy enough to shift the electrons within them to some other definite state, less stable than the normal one. Saha, who has discussed this matter in detail, states that the relative numbers of atoms which are in condition to absorb lines of these various groups probably depend upon the temperature alone—high temperature favoring the absorption of the subordinate lines. It is obvious that "high" temperature must here be a relative term, the scale for any element being roughly proportional to the temperature at which ionization becomes large.

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[1] The existence of such an effect for iodine has been experimentally proved by Smythe and Compton (*Physical Review*, **16**, 501, 1920). Its importance in the solar atmosphere was suggested to the writer, in conversation, by Dr. Lindblad.

The main features of the behavior of the lines of the alkali metals in spots, as Saha has shown, may be accounted for by the relative ease of ionization, sodium being considerably ionized, potassium still more, and rubidium completely so, except in spots. But the fact that lithium, like rubidium, appears only in the spots, cannot be explained on the basis of ionization potentials alone, for it is harder to ionize than sodium, and the percentage of neutral atoms should be greater.

A solution may be found in the probable difference in the relative abundance of the elements. Sodium and potassium are among the most abundant of terrestrial metals, being present in the earth's crust in nearly equal amount (each about 2 per cent of the whole). Lithium is relatively rare on the earth, and, if the proportion present in the solar atmosphere is sufficiently small, it may well be that the number of neutral atoms which remain is insufficient to produce perceptible absorption lines, except in the spots, where the ionization is less. Among the far more numerous sodium atoms the number which escape ionization, though a smaller percentage of the whole, is so great that there are enough of them in a condition to absorb the subordinate series fairly strongly, and the principal series heavily. In potassium the percentage of neutral atoms is much less, and only the lines of the principal series appear, with moderate intensity, but much strengthened in the spots.

It remains somewhat puzzling, however, why rubidium, which is much easier to ionize than lithium, should show nearly the same behavior in the spot spectrum. According to Table III the percentage of neutral rubidium atoms should be less than one-tenth as great as in the case of lithium. The increase in this percentage in spots is great enough for both elements to account for the appearance of their lines there, and not on the disk, provided that we assume that the pressure in the region where the lines are absorbed is 0.01 atmospheres or less (which seems reasonable). But it is hard to see why the rubidium lines should appear at all, unless it is considerably more abundant in the sun than lithium is, which does not at all seem to be the situation on the earth.

Passing to the alkaline earths we find improved opportunities. While the ionized atoms of the alkali metals give no recognized lines in the visible spectrum, those of calcium and its congeners are unique in that the fundamental lines of the ionized atom lie within the region accessible to astrophysics. The great importance of these lines (H and K of calcium and λλ 4078, 4215 of strontium) is thus explained. They enable us to detect the ionization of these easily ionized atoms as soon as it occurs. The ionized atom of calcium is at once capable of absorbing the H and K lines.

From Table III it appears that, under the pressures which probably prevail in the reversing layer, the greater part of the calcium is in the singly-ionized condition.

H and K, therefore, are the lines which the atoms of one of the most abundant constituents of the solar atmosphere absorb most powerfully, when in that state in which under solar conditions they are most likely to occur. There are no other lines in the visible spectrum, so far as known, for which these three maximal conditions are all satisfied.

It is not surprising, therefore, that they are exceptionally conspicuous, and that few other lines play so important a part throughout almost the whole of the sequence of stellar spectra. A great deal of the calcium must be ionized, even in the spots, and this, coupled with the fact that the absorption in the H and K lines is probably far more than saturated, makes it easy to understand why these lines are not perceptibly weakened in the spots. The same remark applies, with even greater force, to the corresponding enhanced lines of strontium and barium.

Magnesium should also be considerably ionized in the reversing layer; but the only line by which we can detect the presence of ionized magnesium (λ 4481) is absorbed only by atoms which have undergone two internal changes subsequent to ionization, corresponding to the shifting of an electron from its normal position to the states denoted symbolically by 2p and 3d, and demanding the absorption of large amounts of energy. The temperature of the reversing layer, compared with that at which a considerable percentage of these Mg+ atoms would lose a second electron, is undoubtedly low. Hence very few such atoms indeed should be in a state to absorb λ 4481, and it is not surprising that it is excessively faint in the solar spectrum, if it appears at all, or that it becomes conspicuous only in the hotter stars.

If the earth's atmosphere transmitted light of short enough wave-length, we would probably find that the fundamental pair of enhanced magnesium lines near λ 2800 were rivals of H and K in intensity and astrophysical importance.

The very strong infra-red lines of calcium at λλ 8498, 8542, 8662, belong to the diffuse series and represent a stage of excitation intermediate between H and K and λ 4481. King[1] has found that they cannot be produced in the furnace, at temperatures at which H and K appear strongly. It may safely be predicted that they are weakened in the spot spectrum, and the matter will be investigated when suitable photographs can be obtained. The corresponding group for strontium is hopelessly far in the infra-red; but the leading pair of the sharp series of enhanced lines is accessible. One line is concealed by a blend; the other is greatly weakened in spots.

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[1] *Mt. Wilson Contr.*, No. 150; *Astrophysical Journal*, **48**, 13, 1918.

For barium, which is easier to ionize a second time, the enhanced lines of the diffuse series are not perceptibly weakened in spots.

The behavior of the lines due to the neutral atoms is also in accordance with this theory. Beginning with zinc, which has the highest ionization potential, we find that the prominent triplet in the blue is greatly weakened in spots, and the line in the red is also weakened, though to a less degree. These lines belong to the sharp series of triplets and the diffuse series of single lines, both of which are absorbed by atoms in which the electron has been shifted one step from its normal position (to 2p or 2P). Measured by a scale appropriate to the zinc atom, the temperature of the photosphere must be low and that of the spots very low; hence the weakening of these subordinate lines is just what should be expected.

It is of interest to note that the lines of hydrogen, which also belong to a subordinate series and to an element for which the ionization potential is high, are much weakened in the spot spectrum.[1]

The lines of magnesium are very little affected in spots. In this case the effects of higher excitation and greater ionization at the higher temperature of the reversing layer appear very nearly to balance one another except in the case of λ 4571, which is a characteristic low-temperature line, and is the easiest of all magnesium lines to excite in the laboratory.[2] This line, as might be expected, is strengthened in the spots, but only to a moderate degree.

For calcium, all the important lines of the neutral atom are strengthened in spots, indicating that the controlling factor is the diminution of ionization at the lower temperature, which increases the number of neutral atoms. Where allowance is made for the general increase toward the red of the degree to which lines are affected in spots, it appears that the strengthening is greatest for the lines absorbed by atoms in which the electron is in the normal state, 1S; less for those of the subordinate series, in which the electron starts from the state 2P (for single lines) or 2p (for triplets); and least of all for those of the Bergmann or combination series in which it starts from 3D or 3d. According to Saha[1] the ratio of the number of atoms which are in any one of these states to the number which are in the preceding stage should increase with the temperature. Hence the effect of the diminished number of neutral atoms in the hotter regions is almost neutralized for the lines of the "outer" series (3D–4F and 3D–2P) and increased for the fundamental lines (1S–2P), and 1S–2p.[2] The first of these, λ 4227, is so far to the violet that the effect in spots is not conspicuous, though great for a line in this region. This is very prominent in the cooler stars. The second, λ 6573, is one of the most remarkable of all sun-spot lines. It is an exact analogue of λ 4571 of magnesium, which likewise shows the greatest strengthening of any line of the element to which it belongs.

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[1] Adams, *Mt. Wilson Contr.*, No. 40, p. 10; *Astrophysical Journal*, 30, 95, 1909.

[2] Foote and Mohler, *Philosophical Magazine*, 37, 40, 1919.

For strontium, the ionization is still greater, and the amount of the element present in the solar atmosphere probably much less, if we may judge by the earth's crust, in which strontium and barium are much rarer than calcium and magnesium. It is not surprising, therefore, that only the strongest lines of the neutral atom appear in the solar spectrum, or that these should be strengthened in the spots, especially the fundamental line λ 4607. λ 6892, which is analogous to λ 6573 of calcium, is absent even in the spot spectrum.

The probable reason for this is that the temperature, even in the spots, is high on the scale which it is necessary to use for the neutral strontium atom, and that, under this high excitation, the proportion of atoms in which the electron undergoes the corresponding change is small.

For barium the final stage appears to have been reached. The lines of the neutral atom do not appear at all either in the spectrum of the sun or of the spots. Even the fundamental line 1S–2P (λ 5535) is gone. This can be explained only on the assumption that barium is completely ionized in the sun, leaving not a trace of the neutral atoms, for the element is abundant enough, as is shown by the strength of its enhanced lines.

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[1] *Proceedings of the Royal Society*, A, **99**, 142, 1921.

From the qualitative standpoint this explanation is quite satisfactory, but a quantitative study brings out difficulties. The ionization potentials for sodium and barium, as determined from their spectral series, should be practically identical. Hence, by the theory developed above, the atoms of both should be ionized to the same degree. But the direct spectroscopic evidence indicates that barium is ionized to a much greater extent than sodium. The absence of the flame lines even in spots, and the fact that the enhanced lines of the diffuse series are not sensibly weakened, would indicate a decidedly lower ionization potential both for Ba and Ba+.

A similar discrepancy, though less marked, exists in the case of strontium.

As the theory of the subject is still in an early stage of development, these facts may be taken as the results of an experiment performed by nature upon a grand scale, which may be used in improving our theories; but speculation as to the exact form of this improvement would at present be premature.

4. GENERAL COMMENTS

The data detailed above cover all the cases in which elements, for which the ionization potentials and spectral series are known, possess lines of any notable strength within the region in which the sun-spot spectrum can be satisfactorily investigated. It is evident that they are all in striking agreement with Dr. Saha's theory of ionization and that they confirm it in a very conclusive fashion, both in the matters in regard to which he has made predictions, and in various others which are here discussed for the first time.

There are many other elements whose lines show conspicuous changes in the spots; but no data are at present available regarding the ionization potentials. Most of these elements have spectra rich in lines, for which the series relations are known very incompletely or not at all.

The behavior of these lines in the spot spectrum,[1] coupled with that in the furnace, arc, and spark, makes it possible to arrange them in a decidedly definite order.

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[1] Adams, *Mt. Wilson Contr.*, No. 40, pp. 6–26; *Astrophysical Journal*, **30**, 91–111, 1909.

A few of the elements already discussed are added for comparison. Lines which are relatively strong at the lowest temperatures (King's Class I) are called flame lines for brevity.

Calcium.—All lines strong in sun. Flame lines much strengthened in spots. Arc lines considerably strengthened in spots. Enhanced lines produced at medium temperature in furnace,[2] very strong in arc, and not weakened in spots.

Scandium.[1]—Flame lines very faint or absent in sun, greatly strengthened in spots. Arc lines faint in sun, much strengthened in spots. Enhanced lines produced in furnace at high temperature, strong in arc, and slightly weakened in spots.

Titanium[2] *and Vanadium*[3]—These elements behave in a very similar fashion, both in the laboratory and in sun-spots. The furnace lines are greatly strengthened in spots and the arc lines strengthened. Practically all the arc lines may be obtained in the furnace. The enhanced lines can be obtained in the furnace only with great difficulty, and are weakened in spots.

Iron,[4] *manganese,*[5] *and chromium*[3].—These again are similar in their behavior. The flame lines are strengthened in the spot spectrum, the arc lines little affected, and the enhanced lines much weakened. Many arc lines are very hard to produce in the furnace, and the enhanced lines cannot be produced there at all, and are faint in the arc. The percentage of lines which are strengthened in the spot spectrum is greater for chromium than for manganese and still smaller in the case of iron.

Cobalt and nickel.[6]—The number of lines weakened in spots is nearly as great as that of the strengthened lines in the case of cobalt, and much greater for nickel. The number of arc lines which cannot be obtained in the furnace is greater for nickel than for cobalt.

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[1] King, *Mt. Wilson Contr.*, No. 214; *Astrophysical Journal*, **54**, 28, 1921.

[2] King, *Mt. Wilson Contr.*, No. 76; *Astrophysical Journal*, **39**, 139, 1914.

[3] King, *Mt. Wilson Contr.*, No. 94; *Astrophysical Journal*, **41**, 86, 1915.

[4] King, *Mt. Wilson Contr.*, No. 66; *Astrophysical Journal*, **37**, 239, 1913.

[5] King, *Mt. Wilson Contr.*, No. 198; *Astrophysical Journal*, **53**, 133, 1921.

[6] King, *Mt. Wilson Contr.*, No. 108; *Astrophysical Journal*, **42**, 344, 1915.

A number of cobalt lines and one nickel line described by Lockyer as enhanced appear in the furnace, but King gives good reason for supposing that these are not enhanced lines of the regular type.

Copper reveals itself in the sun by the first pair of the principal series, λλ 3274, 3247, which are produced when mere traces of the metal are present. They are recorded as of intensity 10 by Rowland, yet no trace of the lines of the subordinate series appears; although λλ 5782, 5700 are furnace lines.[1] The limit of the principal series, 1605 A,[2] corresponds to an ionization potential of 7.7 volts.

For *zinc*, as has already been noted, the arc lines are fairly strong in the sun, but greatly weakened in the spots. Finally, at the end of the series, come *silicon*, for which all the solar lines are greatly weakened in spots, and *oxygen*. The triplet of the latter in the extreme red, λλ 7772–7775, is faintly present in the arc, stronger in the spark, and, though fairly strong in the sun, is completely obliterated in the spot spectrum.[3] Yet the series relations of these lines, which belong to a combination series 2s–mp, show conclusively that they are produced by the neutral atom.

Part of the effect observed in the spots, however, is probably due to the formation of titanium oxide, which must diminish considerably the amount of free oxygen in the solar atmosphere.

The foregoing list shows a definite sequence in the behavior of the elements, alike in furnace, arc, and sun-spots, which suggests strongly that we have to do with a series in which the ionization potential steadily increases (except perhaps in the case of copper, for which the data are very meager). It cannot be a matter of chance that the list, from calcium, and, indeed, from potassium to zinc in order of increasing difficulty of spectroscopic excitation, is a complete list of the intervening elements in the order of their atomic numbers. It appears highly probable that in this region at least the ionization potential increases with the interval in the periodic table which separates the element from the preceding noble gas. Whether this relation holds good more generally cannot be known until further data become available, but it is of interest to note that silicon and oxygen, which are evidently decidedly hard to ionize, fit in very well, since, in the known cases, the ionization potential falls steadily for corresponding elements in each successive period of the periodic table.

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[1] King, *Astrophysical Journal*, 21, 250, 1904.

[2] Konen, *Das Leuchten der Gase* (Braunschweig, 1913), p. 146.

[3] Merrill, *Mt. Wilson Contr.*, No. 183, p. 7; *Astrophysical Journal*, 51, 247, 1920.

The behavior in the spot spectrum of most of the elements in this list is intermediate between that of calcium and that of zinc. It may therefore be surmised that the ionization potentials for these metals lie between six and nine volts, as Saha has suggested[1] without specifying his reasons.

One further comment may be made on the foregoing. It is evident that the enhanced lines of an easily ionized element, such as barium or calcium, may be produced with a weaker excitation than the arc lines of an element of difficult ionization, such as zinc or silicon; the intervening elements present all degrees intermediate between these extremes. The mere fact that an enhanced line appears in a given spectrum is therefore not a proof that the temperature, for example, is high on an absolute scale, but only that it is high relative to the scale appropriate for the element in question. It would probably be fairly safe to assume that temperatures proportional to the various ionization potentials produce similar effects upon different atoms, but further knowledge is necessary before we can be sure of this.

The principles of the ionization theory will evidently be of great importance throughout the whole field of astrophysics, and Dr. Saha[2] has made an application of the highest interest to the question of the physical meaning of the sequence of stellar spectra. The writer hopes to present a discussion of many features of this problem at some time in the not distant future.

The possibilities of the new method appear to be very great. To utilize it fully, years of work will be required to study the behavior of the elements mentioned above and of others, in the stars, in laboratory spectra, and by the direct measurement of ionization, but the prospect of increase of our knowledge, both of atoms and of stars, as a result of such researches, makes it urgently desirable that they should be carried out.

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[1] *Philosophical Magazine*, **40**, 822, 1920.

[2] *Proceedings of the Royal Society*, A, **99**, 135, 1921.

MOUNT WILSON OBSERVATORY
August 1, 1921

NOTE.—An admirable summary of the theory of ionization by Mr. E. A. Milne, in the *Observatory* for September (44, 261, 1921), has appeared since the completion of the foregoing discussion. He points out the influence of the free electrons resulting from the dissociation of easily ionized atoms in diminishing the ionization of other atoms, and gives the correct equation for second-stage ionization. Section 1 of the present paper was prepared independently and carries the analysis somewhat further, so that it appears sufficient to call attention here to Mr. Milne's prior publication.

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THE EXCITATION OF THE ENHANCED SPECTRA OF SODIUM AND POTASSIUM IN A LOW VOLTAGE ARC[1]

By PAUL D. FOOTE, W. F. MEGGERS, and F. L. MOHLER

ABSTRACT

Variation of the spectra of Na and K with energy of electronic impact.—The quantum theory, as developed by Bohr and Sommerfeld, leads to the conclusions: that all the arc lines are emitted by neutral atoms and should be excited at voltages above the ionization voltages, 5.1 for Na and 4.3 for K, and that the enhanced lines are emitted by singly ionized atoms and, in the case of the alkalies, should be excited when (*b*) the valence electron is knocked out of its orbit, after (*a*) an electron from the outer ring (*L*-ring for Na, *M*-ring for K) has been removed. In the case of Na, the ionization (*a*) requires 35 volts if it is complete, giving *La*-radiation, or 30 volts if the electron goes to the *La* orbit; and since the displacement (*b*) requires a maximum of 14 volts, the enhanced spectrum for Na should be excited by two successive impacts at 35 volts and possibly at 30 volts, and the highest frequency should correspond to about 14 volts. The corresponding figures for K are: 23 volts for the excitation of the *Ma*-radiation, possibly 20 volts for *Ma* and the enhanced spectrum, and 11 volts for the energy corresponding to the highest frequency. Since these conclusions contradict the general belief that high potentials are necessary to excite these enhanced spectra, a series of *spectrograms* was made, with a large quartz spectrograph, of the light excited by low voltage electrons in a special cylindrical tube in which the distance between the central tungsten filament, which supplied the electrons, and the accelerating grid was so short and the vapor pressure used was so low (0.1 to 0.2 mm of Hg) that practically all the electrons entered the force-free space surrounding the grid, with the same energy. The voltages used varied from 3.5 to 5000 for Na, and from 3.5 to 40 for K. The results completely verify the theoretical conclusions. Tables and photographs are given which clearly show the *three-stage development* in the spectra of these elements: (1) Single pair stage: 2.1 to 5.1 volts (Na); 1.6 to 4.3 volts (K). (2) Arc spectrum stage: 5.1 to 30 volts (Na); 4.3 to 20 volts (K). (3) Enhanced spectrum stage: above 30 volts (Na); above 20 volts (K). With a heavy current (1 ampere) at 7 volts in K, the pair λ 4642 becomes very prominent, though it has the

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[1] Published by permission of S. W. Stratton, Director, Bureau of Standards.

series notation ($1s-3d$). Its presence cannot be due to incipient Stark effect, and accordingly furnishes an exception to the principle of selection as applied by Sommerfeld. The *wave-lengths in I.A. and the series notations*, when known, are given for 250 lines.

The quantum theory of spectra attributes enhanced lines to ionized atoms. In the case of simply ionized helium the Bohr theory, as generalized by Sommerfeld, not only gives the correct wave-lengths of the observed enhanced lines, but their fine structure as well.

In applying the quantum theory to heavier atoms certain simplifying assumptions must be made in order to carry out the mathematical analysis. Thus Sommerfeld[1] considers the case where the atom has one electron in an outer orbit and where as an approximation the other electrons may be treated as though lying in a circular coplanar orbit. These electrons, as far as their action on the outer orbit is concerned, may be assumed equivalent to a ring of electricity of equal total charge.

Evaluating the kinetic energy of the outer electron, and applying the integral $\int pdq = nh$ both in respect to azimuth and radius as independent variables, in accordance with the quantum theory, Sommerfeld derives the following expression for the variable term of a spectral series, identical in form with the familiar Ritz equation:

$$(m,a) = \frac{N(E/e)^2}{[n_a + n_r + a - \alpha(m,a)]^2} \tag{1}$$

where E is the excess in charge of the core over that of the ring, n_a the azimuthal and n_r the radial quantum numbers characterizing the outer orbits, and a and α constants which are approximately proportional to E and are functions of n_a but not of n_r.

An atom of an alkali metal has one electron in its outer orbit while that of an alkali earth has two, accounting for the monovalent and bivalent properties respectively For the neutral atom of an alkali metal $(E/e)^2 = 1$, and equation (1) represents the variable terms of the arc spectra. If one of the outer electrons of an atom of an alkali earth is permanently removed, that is, if the atom is simply ionized, equation (1) likewise applies except that $(E/e)^2 = 4$ and the values of a and α are altered in a definite manner.

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[1] *Atombau* (2d ed.), p. 506.

Hence if the enhanced spectrum of magnesium is due to the simply ionized atom, it should resemble the arc spectrum of sodium. Such is the case, even to the appearance approximately of $4N$ and $a^* = 1.5\,a$ instead of N and a in the spectral formulae representing the enhanced lines.[1] A similar relation exists between the spectra of the other elements of these two groups. From the above and further considerations Kossel and Sommerfeld[2] concluded that in so far as enhanced spectra arise in *simply* ionized atoms the enhanced spectrum of any element should resemble the arc spectrum of the element of next lower atomic number. Very likely higher types of enhanced spectra may be produced by atoms which have suffered multiple ionization.

Now, as is well recognized, the complete arc spectrum of any monatomic vapor may be excited by electronic collision with a normal atom when the energy of the impacting electron is eV, where V is the ionization potential. Furthermore for all monatomic vapors not exhibiting metastable modifications[3] the voltage V is related to the highest convergence frequency ν of the arc spectrum by the quantum condition $ch\nu = eV$. The highest frequency in the arc spectra of the alkali earths is $1S$ and of the alkali metals $1s$, corresponding to ionization potentials from 4 to 10 volts. Similarly in the enhanced spectrum of an alkali earth, the highest frequency is $1\mathfrak{S}$, and accordingly at the corresponding voltage V^* the complete simple enhanced spectrum should be excited. We have verified this experimentally for magnesium.[4]

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[1] Sommerfeld neglected to consider the shrinkage of the ring when the number of electrons in the ring is maintained constant while the charge on the core is increased by one unit. If this is taken into account we find for sodium and potassium $a^* = 1.5\,a$ instead of $a^* = 2\,a$ given by Sommerfeld. The former relation is well satisfied if we assign values of 1, 2, 3, etc., instead of 1.5, 2.5, etc., to m in the (ms) terms. Cf. *Journal of the Optical Society of America and Review of Scientific Instruments*, January, 1922.

[2] *Verh. d. Phys. Ges.*, **21**, 240, 1919.

[3] From spectroscopic and other considerations there appear to be two modifications of helium, possibly one in which the two electrons are in crossed orbits and the other in which the orbits are coplanar. In polyatomic vapors the ionization potential may involve the work of dissociation in addition to the ionization of one or more atoms.

[4] The enhanced spectrum of magnesium is produced by electronic impact of 14.97 volts corresponding to the wave-number $1\mathfrak{S} = 121{,}270$, in which case the impact occurs with a simply ionized atom, with the result that the second valence electron

These voltages again are all small, of the order of 10 to 20 volts for the elements in this group of the periodic table.

The simple enhanced spectra of the alkali metals should resemble the complicated arc spectra of the rare gases. Since only recently attempts to correlate the latter in series[1] have been successful it is not surprising that nothing is known of series relations in the enhanced spectra of the alkali metals.

It is possible, however, to estimate roughly the voltages at which the enhanced spectra of sodium and potassium should be excited. In the case of sodium we have the K-ring with 2 electrons, the L-ring with 8 electrons, and the outer ring with the valence electron. There are two direct ways in which the atom may be ionized. First, the valence electron may be removed requiring an amount of work equivalent to 5.1 volts;[2] then a second collision of X volts may expel an electron from the L-ring. Secondly, an electron may be ejected from the L-ring requiring 35 volts,[3] followed by a collision of Y volts expelling the valence electron. X must be somewhat greater than 35, but, since the orbit of the valence electron is much larger than that of the L-ring, it is not evident that the repulsive force of the outer electron can materially assist in the expulsion of an electron from the L-ring. Hence to doubly ionize sodium by the first method should require electronic impacts of maximum value $La+\delta La$ volts where δLa is a small quantity and δ is a factor. The total work of double ionization is then $(La+\delta La+5)$ volts. In the second method of ionization accordingly $Y=\delta La+5$ which must be less than 35 volts. Hence by the first method of excitation, the enhanced spectrum of sodium should appear at $35(1+\delta)$ volts and by the second method at 35 volts, and in both cases should accompany its L-radiation.

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is removed. This paper, *Phil. Mag.*, **42**, 1002, December, 1921, is of especial interest since De Gramont and Hemsalech, *C.R.*, **173**, 505, 1921, have just published a report of experiments in which they found that a potential gradient of 500 volts/cm was required to excite the enhanced spectrum of magnesium.

[1] For example, neon: Paschen, *Annalen der Physik*, **60**, 405, 1919.

[2] Tate and Foote, *Phil. Mag.*, **36**, 64, 1918.

[3] Mohler and Foote, "Soft X-Rays from Arcs in Vapors," *Journal of the Optical Society of America*, **5**, 328, 1921. For sodium $La=35$ volts.

Similar considerations for potassium show that its enhanced spectrum should be excited by direct impact at about 20 volts and should accompany its M-radiation.[1]

It is possible to arrive at an approximate estimate of the value of δ in the foregoing formulae. After expulsion of an electron from the L-ring, the ring shrinks, and provided the valence electron does not fall into the L-ring—a condition which would curtail the emission of the enhanced lines—it assumes a new orbit where the angular momentum is unity. This orbit represents a higher energy level. Following Sommerfeld's method of derivation except that two coplanar rings of electrons are assumed and their shrinkage is given consideration, we find for sodium and potassium:

$$\left.\begin{array}{ll} a^{*}=\dfrac{2(r+x)}{r+y}a & x=\dfrac{112}{(9-s_7)^2} \\ r=\dfrac{Z-9}{(Z-s_{Z-9})^2} & y=\dfrac{128}{(9-s_8)^2} \end{array}\right\} \tag{2}$$

where a^* and a are the values of a in the Ritz formula, equation (1), for the enhanced and arc lines respectively, Z the atomic number and s_n the nuclear defect of the ring, i.e., effect of the interaction of the electrons. To an approximation, omitting terms beyond a in the Ritz formula:

$$V=\frac{13.55}{(1+a)^2} \tag{3}$$

$$V^{*}=\frac{54}{(1+a^{*})^2} \tag{4}$$

where V is the simple ionization potential and V^* the voltage corresponding to the work required to remove the valence electron from its new quantized orbit, after the removal of an electron from the L-ring. V^* accordingly corresponds to the highest convergence frequency in the spark spectrum.

Equation (2) gives $a^* = 1.5\ a$ for sodium and potassium. Computing a from equation (3) and the known ionization potentials, we obtain from (4) the following:

$V^* = 14$ for sodium

$V^* = 11.5$ for potassium.

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[1] Mohler and Foote, *loc. cit.*, observed two M-limits for potassium one at 20 volts and one at 23 volts.

and plate have, at the instant of collision, a velocity corre
to the impressed field. The gas pressure registered by
was less than 0.001 mm of mercury and the vapor pre
of the order 0.1 to 0.2 mm of mercury.

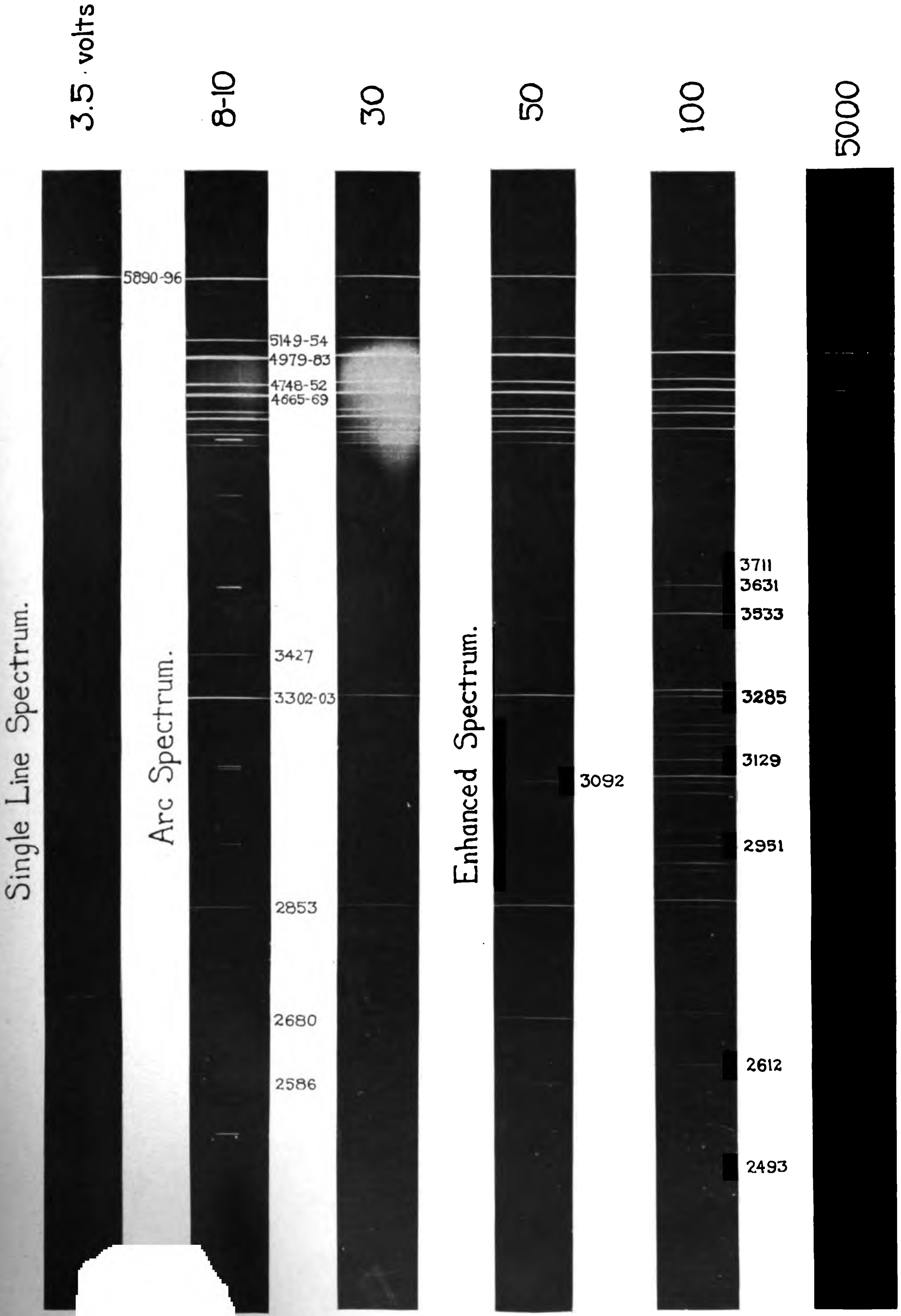
The arcs at various exciting voltages were photograp
a large Hilger Type C quartz-spectrograph. Plate II
results for sodium and Plate III for potassium. Ordinar
plates were used for all exposures except the 3.5 volt pota
in which case the plate was made red sensitive by b
dicyanin. The pair λ 7699–7664, although present in th
at the higher voltages, was not recorded by the "blue-
plates used for the other spectrograms. The lines on all
were carefully measured and reduced to wave-length
method of the Hartmann-Cornu interpolation formula, u
of a mercury comparison spectrum and known arc and s
as normals.

SODIUM

Plate II shows the three successive stages in the sodi
trum. The single-line spectrum consisting of the pair
is excited above the resonance potential, 2.1 volts, and
ionization potential 5.1 volts. From 5.1 volts to about
the arc lines appear alone. At 30 volts numerous sp
begin to appear while above 40 volts the complete
spectrum is excited.

The lower spectrogram was obtained by connectin
5000-volt transformer between the hot wire and grid. T
was too rare for a discharge to pass except when the ca
heated to give a copious emission of electrons. Since th
resistance of the discharge gap must have been small
to the impedance of the transformer, the actual applied v
undoubtedly much less than 5000 volts. Very few
lines not excited at 30 volts were observed in this source
intensity was considerably greater. All of the spectr
sodium were made with a very narrow slit and much of
readily apparent on the negatives is obscured in the rep
On this account reference must be made to Table I f
careful consideration of the excitation of the enhanced
verification of the foregoing statements.

PLATE II



SPECTRA OF SODIUM

The total work of double ionization is accordingly:

Total work $= La + V^* = 35 + 14 = 49$ volts for sodium
$= 23 + 11 = 34$ volts for potassium.

Hence for sodium

$$La + \delta La + V = 49$$
$$\delta La = 9$$
$$\delta = 0.3.$$

Similarly for potassium

$$\delta = 0.3.$$

These values of δ confirm the estimates made above from more qualitative considerations. It is of interest to note that in sodium, for example, although it requires but 14 volts to excite the enhanced spectrum, this spectrum cannot be produced at this voltage since as a prerequisite the displacement of an electron from the L-ring is essential, which necessitates electronic impact of higher energy value. This is in contradistinction to the phenomena observed in the case of the alkali earths. Here the removal of the valence electron at the simple ionization potential leaves the atom in the fundamental state for the emission of enhanced lines (just as the normal, unexcited atom is in the fundamental state for the emission of arc lines) and further impacts of even less energy value are sufficient to excite the "single-line" *enhanced* spectrum $1\mathfrak{S} - 2\mathfrak{P}$. However, with the alkalis, the simple ionization does not leave the atom in the fundamental state for the emission of enhanced lines, but rather in the state for the emission and even absorption of La and other X-ray lines.

In the X-ray spectra of elements of higher atomic number it is known that $K\alpha$ cannot be produced except when the limit $K\alpha$ is reached, under which condition all K-lines appear simultaneously. If a similar condition held for the L-radiation of sodium, it would be impossible to excite the enhanced spectrum below the voltage corresponding to the limit $La = 35$ volts. However, although a single-line X-ray spectrum has never been observed, it must be a possible phenomenon for elements of low atomic number where the ring concerned is the outermost "saturated" orbit. Hence $La = 35 - V = 30$ volts might be produced at 30 volts instead of at

La=35. Such being the case the enhanced spectrum could be excited by successive impact at the voltage corresponding to *La*, the first impact (or equivalent absorption of radiation) displacing the electron to the *La* orbit, where a second impact is more than sufficient to doubly ionize. Still further, Mohler and Foote observed a new series of critical potentials of lower value than *La*, for several of the elements of low atomic number, which may be related to the *L*-series. If so, the enhanced spectra of sodium might be excited by successive impact in an arc at the voltage corresponding to this new limit.

In view of the fact that heretofore all investigators have employed and believed requisite an exciting field of many thousand volts for the production of the enhanced spectra of sodium and potassium, the low values above estimated necessitated experimental verification.

The form of the discharge tube employed is illustrated by Figure 1. Its novelty consists in the use of a grid (helical coil) mounted extremely close to a tungsten hot wire cathode (or in some cases an equipotential surface heated inside) and in metallic contact with a concentric hollow cyclinder of relatively large diameter. By properly regulating the temperature of the sodium or potassium vapor, a vapor pressure may be maintained for which relatively few electronic-atomic collisions occur over the short accelerating field between the cathode and grid, and the majority of electrons falling into the large force-free space between the grid

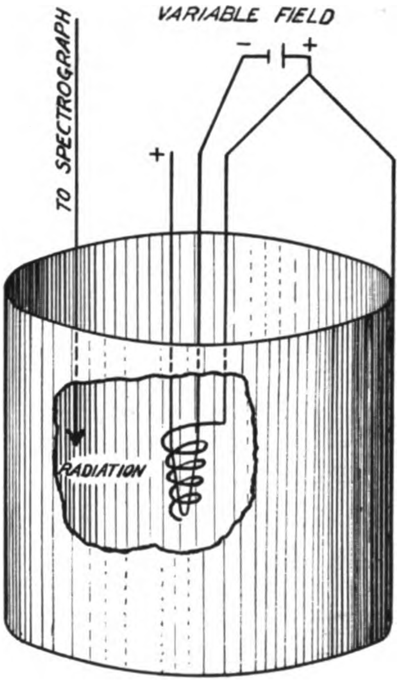


FIG. 1.—Discharge tube. The electrons attain the full velocity of the impressed field.

and plate have, at the instant of collision, a velocity corresponding to the impressed field. The gas pressure registered by the gage was less than 0.001 mm of mercury and the vapor pressure was of the order 0.1 to 0.2 mm of mercury.

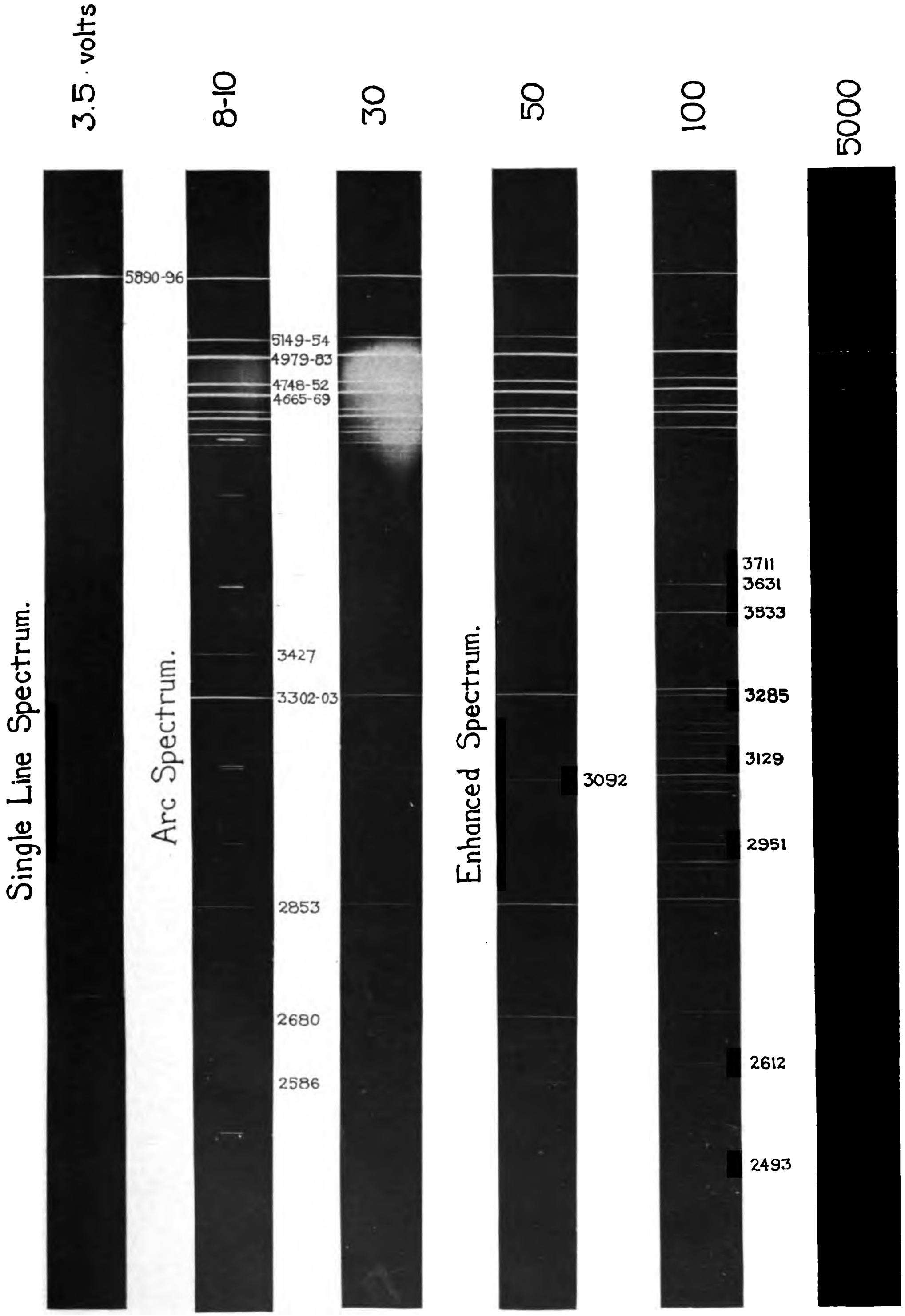
The arcs at various exciting voltages were photographed with a large Hilger Type C quartz-spectrograph. Plate II shows the results for sodium and Plate III for potassium. Ordinary Seed 30 plates were used for all exposures except the 3.5 volt potassium arc in which case the plate was made red sensitive by bathing in dicyanin. The pair λ 7699–7664, although present in the spectra at the higher voltages, was not recorded by the "blue-sensitive" plates used for the other spectrograms. The lines on all the plates were carefully measured and reduced to wave-lengths by the method of the Hartmann-Cornu interpolation formula, using lines of a mercury comparison spectrum and known arc and spark lines as normals.

SODIUM

Plate II shows the three successive stages in the sodium spectrum. The single-line spectrum consisting of the pair D_1 and D_2 is excited above the resonance potential, 2.1 volts, and below the ionization potential 5.1 volts. From 5.1 volts to about 30 volts the arc lines appear alone. At 30 volts numerous spark lines begin to appear while above 40 volts the complete enhanced spectrum is excited.

The lower spectrogram was obtained by connecting a small 5000-volt transformer between the hot wire and grid. The vapor was too rare for a discharge to pass except when the cathode was heated to give a copious emission of electrons. Since the effective resistance of the discharge gap must have been small compared to the impedance of the transformer, the actual applied voltage was undoubtedly much less than 5000 volts. Very few enhanced lines not excited at 50 volts were observed in this source but their intensity was considerably greater. All of the spectrograms of sodium were made with a very narrow slit and much of the detail readily apparent on the negatives is obscured in the reproduction. On this account reference must be made to Table I for a more careful consideration of the excitation of the enhanced lines, and verification of the foregoing statements.

PLATE II



SPECTRA OF SODIUM

TABLE I

SPECTRA OF SODIUM

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
2318.0	1−	1−				
2475.51					1−	$1s-10p_{1,2}$
90.61				1−	1−	$1s-9p_{1,2}$
2493.36	4	4	2	1		
2497.3	2			1		
2506.40	1	1				
12.14	1−	1−	1−	1	1	$1s-8p_{1,2}$
15.50	1	1				
31.54	3	2	1			
43.80	1−	1	1	2	2	$1s-7p_{1,2}$
86.27	1	1				
93.89	1	2	2	3	3	$1s-6p_{1,2}$
2595.0	1−	1−				
2612.18	5	4	1	1		
61.72	4	3	1	1		
71.94	5	3	1	1		
78.2	3	2	1			
2680.37	1	3	3	4	3	$1s-5p_{1,2}$
2809.76	4	3	1	1		
18.5	1	1				
30.0	1	1				
39.36	1	1				
41.99	6	4	2			
52.80	3	4	4	5	5	$1s-4p_{1,2}$
59.56	3	2	1			
61.2	1−	1−				
70.89 71.39	3	2	1			
73.0	1−	1−				
81.33	4	3	2	1		
86.46	3	2	1			
2894.15	4	3	1	1		
2901.39	3	2	1			
05.16	6	4	3	2		
17.76	4	3	3	2		
19.46	3	3	2	1		
21.14	3	3	2	1		
23.65	2	2	1			
31.44	1−	1−				
34.42	1	1				
2938.11	3	2	1			
45.83	2	1				
47.51	3	2	1			
51.53	8	5	2	2		
60.10	1	1	1−			

TABLE I—*Continued*

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
71.1	1—	1—	....	....	....	....
75.20	5	3	2	1	....	....
77.52	1	1	....	....	....	....
79.92	7	4	2	1	....	....
2984.44	6	4	2	1	....	....
3007.71	2	2	1	....	....	....
15.81	3	2	1	1	....	....
29.70	3	3	1	1	....	....
37.29	4	3	1	1	....	....
50.48	3	1	1—	....	....	
53.97	4	3	1	....	....	....
56.34	7	5	3	2	....	....
61.75	1—	....	....	....	....	....
64.8	2	1	1—	....	....	....
66.74	2	2	1—	....	....	....
74.63	4	3	1	1	....	....
78.51	5	4	3	....	....	....
80.58	2	2	....	....	....	....
92.91	12	8	6	3	....	....
3104.6	2	2	1	....	....	....
07.6	1—	1—	....	....	....	....
24.63	2	2	1—	....	....	....
29.57	7	5	4	2	....	....
35.65	5	4	3	1	....	....
38.11	1	1	....	....	....	....
46.07	1	2	....	....	....	....
49.48	5	4	3	1	....	....
64.11	7	5	2	2	....	....
75.38	1	1	....	....	....	....
79.19	3	2	....	....	....	....
3189.94	7	5	3	2	....	....
3212.42	7	5	3	2	....	....
16.5	1	1	....	....	....	....
26.1	2	2	....	....	....	....
35.1	3	2	1—	....	....	....
51.3	1	1	....	....	....	....
58.36	7	4	2	1	....	....
60.4	1	1	....	....	....	....
74.27	3	2	....	....	....	....
85.76	8	7	4	2	....	....
3302.33	3	4	3	4	10	$1s-3p_1$
02.93	2	3	2	3	8	$1s-3p_2$
05.2	2	2	....	....	....	....
18.2	3	2	....	....	....	....
20.7	1—	1	....	....	....	....

TABLE I—*Continued*

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
3327.9	3	2	....	....	....	....
3400.2	1−	1−	....	....	....	....
27.1	1	1	1	4	7	$1s-3d$
3462.58	2	2	1	....	....	....
3533.08	12	8	5	3	....	....
3631.31	5	4	2	1	....	....
34.3	1	1	....	....	....	....
3711.15	4	3	1	....	....	....
3745.6	1−	1−	....	....	....	....
3881.0	1	1−	....	....	....	....
4124.0	2	2	....	....	....	....
4141.0	....	....	....	....	1−, *d*	$2p-18d$
48.0	....	....	....	....	1−, *d*	$2p-17d$
56.0	....	....	....	....	1−, *d*	$2p-16d$
65.0	....	....	....	....	1−	$2p_2-15d$
67.8	....	....	....	....	1−	$2p_1-15d$
77.4	....	....	....	....	1−	$2p_2-14d$
80.2	....	1−	....	1−	1	$2p_1-14d$
92.6	....	....	....	....	1−	$2p_2-13d$
4195.5	....	1	....	1	1	$2p_1-13d$
4212.4	....	....	....	....	1	$2p_2-12d$
15.4	....	1−	1−	1	2−	$2p_1-12d$
22.8	....	....	....	....	1, *d*	$2p-11s$
38.6	1−	....	....	....	1	$2p_2-11d$
41.6	1−	1	1	1	2	$2p_1-11d$
49.4	....	....	....	....	1−	$2p_2-10s$
52.4	....	1−	1−	1−	1−	$2p_1-10s$
73.2	1−	....	....	....	2	$2p_2-10d$
4276.3	1	1	1	1	3	$2p_1-10d$
87.5	....	1−	1−	1−	1−	$2p_2-9s$
4290.6	1−	1	1	1	1	$2p_1-9s$
4321.1	1	1	1	1	3	$2p_2-9d$
24.3	2	2	2	2	4	$2p_1-9d$
41.3	....	1	1	1	1	$2p_2-8s$
44.5	1	1	1	1	2	$2p_1-8s$
90.14	3	2	2	2	4	$2p_2-8d$
4393.45	4	3	3	3	6	$2p_1-8d$
4419.94	1−	1	1	1	2−	$2p_2-7s$
23.31	1	2−	2−	2−	3	$2p_1-7s$
94.266	5	4	4	4	5	$2p_2-7d$
97.724	6	5	5	5	7	$2p_1-7d$
4541.671	2	1	1	1	2	$2p_2-6s$
4545.218	3	2	2	2	4	$2p_1-6s$
4664.858	7	4	4	4	7	$2p_2-6d$
4668.597	9	6	6	6	9	$2p_1-6d$

TABLE I—*Continued*

Wave-Length	5000 Volts	100 Volts	50 Volts	30 Volts	10 Volts	Notation for Arc Lines
4748.016	4	2	2	2	3	$2p_2-5s$
4751.891	5	2	2	2	4	$2p_1-5s$
4978.608	9	6	6	7	9	$2p_2-5d$
4982.864	10	8	8	9	10	$2p_1-5d$
5149.090	2	2	2	2	5	$2p_2-4s$
5153.645	3	3	3	3	6	$2p_1-4s$
5682.675	1	1	1	1	2	$2p_2-4d$
5688.222	1	1	1	1	2	$2p_1-4d$
5889.965	15	10	10	8	15	$1s-2p_1$
5895.932	10	8	8	6	10	$1s-2p_2$
6154.214	1	....	....	....	1	$2p_2-3s$
6160.725	1	....	....	....	1	$2p_1-3s$

DATA FOR TABLE I

Exposure Number	Volts	Milliamperes	Time of Exposure
23	5000	....	4 hours
16	100	150	30 min.
17	50	40–140	50 min.
21	30	65	65 min.
20	8–10	2000	125 min.

The wave-lengths in Table I are in international angstrom units. The spark lines given by Schillinger,[1] having been corrected from the Rowland to the international scale, are represented by six significant figures while the values for the arc lines from λ 4390.14 to λ 6160.725 are those recently measured by Datta.[2] The remaining values are given to four or five figures and are the means of several determinations from the quartz prism spectrograms; they are probably not in error by more than one unit in the last place. Approximate values for the wave-lengths of many subordinate series lines were given by Zickendraht[3] and our numbers are in fair agreement with his for the higher terms.

It is seen from this table that the critical voltage for the excitation of the enhanced spectrum of sodium is about 30 volts, in the neighborhood of the values for *La* and *La*.

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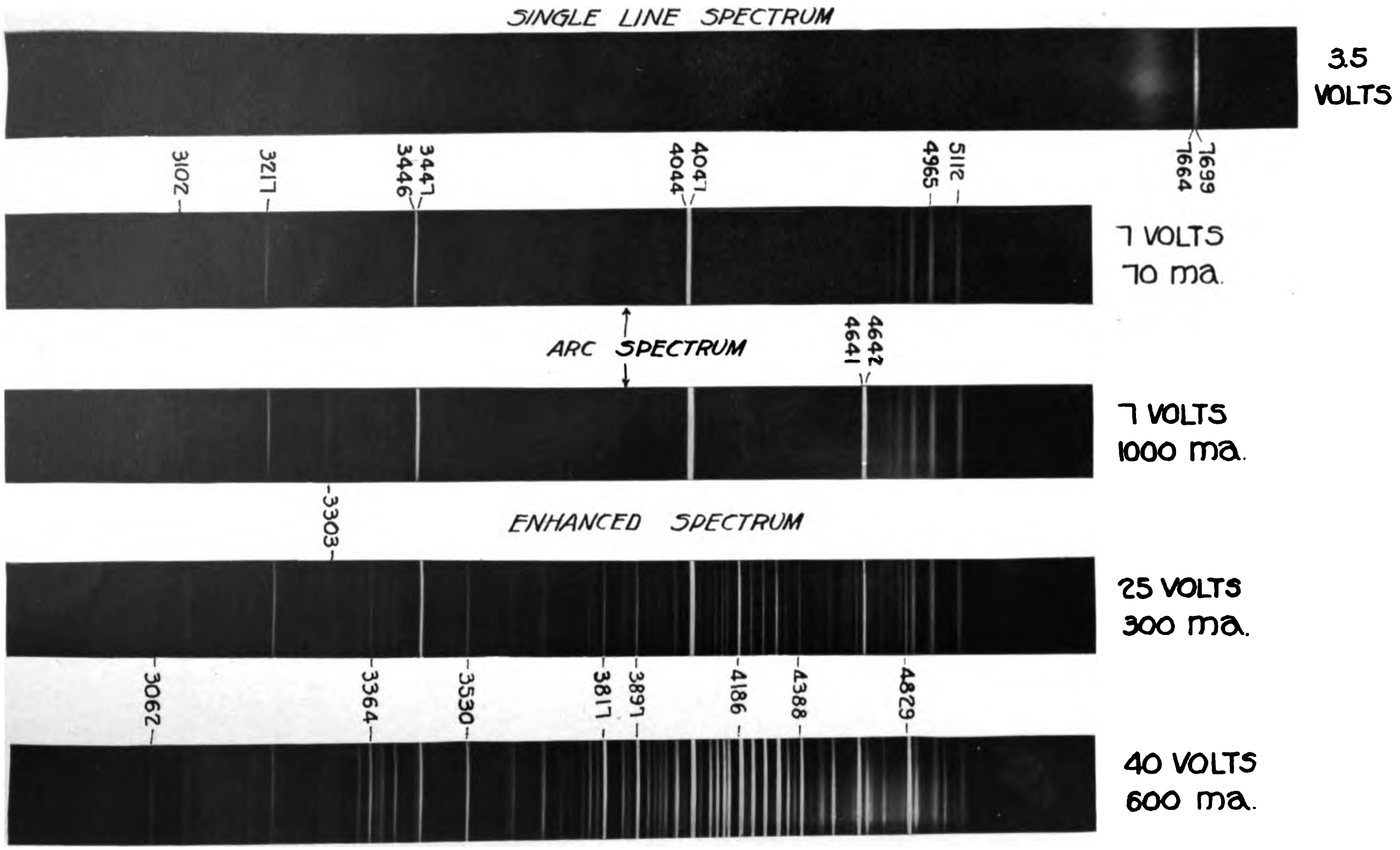
[1] Schillinger, *Sitzungsberichte der Kais. Akad. d. Wiss.*, **118**, 605, 1909.

[2] Datta, *Proc. Roy. Soc.* (A), **99**, 69, 1921.

[3] *Annalen der Physik*, **31**, 233, 1910.

PLATE III



Spectra of Potassium

POTASSIUM

Plate III shows the three stages in the development of the potassium spectrum. Table II contains the wave-lengths and estimated intensities of the lines at various exciting voltages from 3.5 to 40 volts. From the resonance potential, 1.6 volts, to the ionization potential, 4.3 volts, the single-line spectrum is emitted, consisting of the pair λ 7699 and λ 7664. Above the ionization potential the complete arc spectrum is excited while in the neighborhood of 20 volts (viz., near *Ma* and *Ma*) the enhanced lines are prominent. A wider slit was used for this series of exposures than for sodium and the results are therefore more favorable for reproduction although less advantageous for precise measurement of wave-lengths. The values for the wave-lengths in Table II are compiled from the measurements of Shillinger and Datta. Only the stronger spark lines are listed, as these serve sufficiently to show how the intensity of the enhanced spectrum rapidly increases when the critical voltage is exceeded. A great many faint spark lines were observed in addition to those given by Shillinger, a fact also noted by McLennan,[1] but their wave-lengths are not known with sufficient accuracy to justify publication.

While the main issue of the present paper concerns the excitation of enhanced spectra, the second and third spectrograms of Plate III, showing the arc spectrum of potassium, are of considerable interest. Both of these exposures were made at 7 volts, the only difference being that an ionization current of 70 milliamperes was employed for the upper spectrogram and 1000 milliamperes for the lower. In the latter case the pair λ 4642 and λ 4641 appears as one of the most prominent lines in the spectrum. This pair has the notation $\nu = 1s - 3d$, representing an interorbital transition where the change in azimuthal quantum number is according to Sommerfeld two units.[2] It accordingly contradicts the Bohr or Rubinowicz principles of selection as applied by Sommerfeld to spectra of the non-hydrogenic type While the line

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[1] *Proc. Roy. Soc.* (A), 100, 182, 1921.

[2] In a paper by Foote, Mohler, and Meggers, to appear in the *Philosophical Magazine*, this and other exceptions to the principle of selection are considered in more detail.

TABLE II

SPECTRA OF POTASSIUM

Wave-Length	40 Volts	25 Volts	20 Volts	16 Volts	6 Volts	3.5 Volts	Notation for Arc Lines
2992.21	....	....	....	1−	1−	....	$1s-8p_{1,2}$
3034.8	....	1	1	1	1	....	$1s-7p_{1,2}$
3062.44	3	2	2−	....	....	....	....
3102.03 02.25	1	3	3	2	2	....	$1s-6p_1$ $1s-6p_2$
05.31	2	1	1	....	....	....	....
29.39	1	1−	1−	....	....	....	....
69.89	1	1	1−	....	....	....	....
87.91	2	1	1	....	....	....	....
3190.42	2	1	1	....	....	....	....
3201.96	1	1−	....	....	....	....	....
17.01 17.50	4	5	5	5	5	....	$1s-5p_1$ $1s-5p_2$
20.80	1	1	....	....	....	....	....
3290.93	2	2−	1	....	....	....	....
3312.63	2	1	1−	....	....	....	....
45.7	5	4	2	....	....	....	....
58.33	1	1	....	....	....	....	....
60.16	1	1	....	....	....	....	....
63.25 64.56	6	5	3	....			
73.84	2	2	1	....	....	....	....
80.97	2	2	1	....	....	....	....
3385.24	2	2	1	....	....	....	....
3404.57	5	4	2	....	....	....	....
22.26	1	1	1−	....	....	....	....
33.61	1	1−	....	....	....	....	....
40.36	2	1	1−	....	....	....	....
46.722 3447.701	5	6	8	8	8	....	$1s-4p_1$ $1s-4p_2$
3530.83	5	5	3	1	....	....	....
3609.26	2	1	1−	....	....	....	....
18.43	3	2	1	....	....	....	....
68.91	1	....	....	....	....	....	....
76.01	2	1	....	....	....	....	....
3681.52	4	3	2	....	....	....	....
3716.51	1	1	....	....	....	....	....
21.93	1	1	....	....	....	....	....
39.13	1	1−	....	....	....	....	....
44.62	1−	1−	....	....	....	....	....
67.37	3	2	1	....	....	....	....
3783.19	4	3	2	1−	....	....	....
3800.60	3	2	1	....	....	....	....
17.99	6	5	3	1	....	....	....
61.87	4	2	1	....	....	....	....

TABLE II—*Continued*

Wave-Length	40 Volts	25 Volts	20 Volts	16 Volts	6 Volts	3.5 Volts	Notation for Arc Lines
73.78	1	1	....	....	....	....	....
3878.31	1	....	....	....	....	....	....
3897.86	7	5	3	2	....	....	....
3923.65	3	2	1—	....	....	....	....
42.86	3	2	1	....	....	....	....
55.27	3	2	1	....	....	....	....
66.75	2	1	1	....	....	....	....
72.61	2	1	....	....	....	....	....
3995.08	2	1	....	....	....	....	....
4001.20	5	3	2	1	....	....	....
12.07	1	....	....	....	....	....	....
25.20	1	....	....	....	....	....	....
44.140	6	7	8	8	8	....	$1s-3p_1$
47.201	4	5	6	6	6	....	$1s-3p_2$
57.95	3	1	....	....	....	....	....
4114.96	5	4	2	1—	....	....	....
34.72	5	4	2	1	....	....	....
49.24	5	4	2	1	....	....	....
4186.23	9	8	4	2	....	....	....
4210.21	2	1	1	....	....	....	....
22.99	5	4	2	1	....	....	....
25.60	5	4	2	1	....	....	....
4263.32	6	5	2	1	....	....	....
4305.01	5	4	2	1	....	....	....
09.08	7	6	4	2—	....	....	....
4388.13	5	4	2	1	....	....	....
4423.72	1	....	....	....	....	....	....
4466.66	2	....	....	....	....	....	....
4505.39	4	2	1	....	....	....	....
4608.31	4	2	....	....	....	....	....
41.585 42.172	2	4	5	4	8	....	$1s-3d'$ $1s-3d$
4659.58	1	....	....	....	....	....	....
4741.6	....	....	....	....	1—	....	$2p_2-10s$
45.58	....	1	1	1	1	....	$2p_2-11d$
54.5	....	....	....	....	1—	....	$2p_1-10s$
59.31	....	1	1	1	1	....	$2p_1-11d$
86.89	....	1	1	1	1	....	$2p_2-9s$
4791.08	....	1	1	1	2	....	$2p_2-10d$
4800.16	....	1	....	1	1	....	$2p_1-9s$
05.19	....	1	1	1	2	....	$2p_1-10d$
28.99	5	3	2	....	....	....	....
49.88	1—	1	1	2	2	....	$2p_2-8s$
56.03	1	2	2	3	3	....	$2p_2-9d$
63.61	1—	1	1	2	2	....	$2p_1-8s$

TABLE II—*Continued*

Wave-Length	40 Volts	25 Volts	20 Volts	16 Volts	6 Volts	3.5 Volts	Notation for Arc Lines
4869.70	1	2	2	3	3	....	$2p_1-9d$
4941.964	1	2	2	2	2	....	$2p_2-7s$
43.02	1	....	....	....	....	....	....
50.816	2	3	3	3	3	....	$2p_2-8d$
56.043	1	2	2	2	2	....	$2p_1-7s$
4965.038	2	3	3	3	3	....	$2p_1-8d$
5005.34	2	....	....	....	....	....	....
55.78	1	....	....	....	....	....	....
84.212	2−	3−	3−	3−	4−	....	$2p_2-6s$
97.144	2	3	3	3	4	....	$2p_2-7d$
5099.180	2−	3−	3−	3−	4−	....	$2p_1-6s$
5112.204	2	3	3	3	4	....	$2p_1-7d$
5323.228	2−	3−	3−	4−	4−	....	$2p_2-5s$
39.670	2−	3−	3−	4−	4−	....	$2p_1-5s$
42.974	2	3	3	4	4	....	$2p_2-6d$
5359.521	2	3	3	4	. 4	....	$2p_1-6d$
....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....
7664.94	....	....	....	....	....	6	$1s-2p_1$
7699.01	....	....	....	....	....	4	$1s-2p_2$

DATA FOR TABLE II

Exposure Number	Volts	Milliamperes	Time of Exposure
2	40	500–600	10 min.
7	25	0–300	32 min.
5	20	100–700	15 min.
8	16	0–500	20 min.
18	6	1200	15 min.
26	3.5	12	7 hours

is well known, its excitation has been thought to be occasioned by the presence of the electrostatic field—an incipient Stark effect.[1] However, in the present case, with the special form of discharge tube employed, the radiation was observed only in the space shielded from the applied electrostatic field—itself only 7 volts. On referring to Table I and Plate II, it will be noted that the corresponding line λ 3427 appears in sodium. Its intensity increases very rapidly as the ionization current is increased, independently of the voltage, provided this exceeds the ionization potential.

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[1] Sommerfeld, *Annalen der Physik*, 63, 249, 1921.

The importance of this result has been pointed out in the paper referred to above, in which we suggest that the explanation may involve a reconsideration of the method whereby single azimuthal quantum numbers have been assigned to each of the *s*, *p*, *b*, and *d* terms.

The following table summarizes the values of the critical voltages characterizing the three-stage development in the spectra of sodium and potassium.

Element	Volts	Notation	Spectrum
Sodium...........	2.09	$1s-2p$	Single pair
	5.11	$1s$	Arc lines
	35–30	....	Enhanced lines and *L*-radiation, $La=35$, $La=30$
Potassium.........	1.60	$1s-2p$	Single pair
	4.32	$1s$	Arc lines
	23 20	....	Enhanced lines and *M*-radiation

Highest frequency in enhanced spectrum: sodium = 14 volts; potassium = 11 volts. Total work of double ionization: sodium = 49 volts; potassium = 34 volts.

Bureau of Standards
December 1, 1921

THE VACUUM-SPARK SPECTRA OF THE METALS[1]

By EDNA CARTER

ABSTRACT

Vacuum-spark spectrograms of Ca, Mg, Cd, Ti and Fe were obtained *with comparison arc and spark spectra*, on portrait films, using for the most part a 20-cm concave grating spectrograph. The Fe spectrum was also photographed with a dyed film from λ 4000 to λ 6600 A. The new spectra show no striking new characteristics. In the case of Ca and also of Mg (see Plate IV), the vacuum-spark spectrum is practically identical with the spark spectrum in air; in the case of Cd (see Plate V), the arc lines are somewhat more intense; with Ti this tendency to strengthen the arc lines is even greater, and the vacuum-spark spectrum of Fe is more like the arc spectrum than the spark. In general these spectra resemble the luminescence spectra produced by cathode-ray bombardment in a high vacuum, and it is probable that the conditions of emission are very similar in the two sources.

I. INTRODUCTION

The production of a brilliant spark between electrodes very close together in a high vacuum was observed by Rowland[2] and also by Wood.[3] The nature of the discharge was studied by Loving.[4] He concludes in regard to the spectrum produced that it is characteristic of the anode, the cathode having no effect, and that it is not analogous to either the spark or the arc. The spectra which he obtained were not sufficiently intense for an exact determination of their character.

Millikan[5] developed independently a method of producing this discharge; and, by using it as a source in a vacuum spectrograph, succeeded with his co-workers in extending the spectra of various elements into the extreme ultra-violet.

It was of interest to see whether a careful study of this source would disclose any peculiarities of metallic spectra in the usual photographic region. A comparison of the vacuum-spark spectra with the arc and the spark spectra of typical metals was therefore undertaken.

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[1] *Contributions from the Mount Wilson Observatory*, No. 219.

[2] Rowland, *Physical Papers*, p. 574.

[3] *Physical Review*, **5**, 1, 1897.

[4] *Astrophysical Journal*, **22**, 285, 1905.

[5] *Ibid.*, **52**, 47, 286, 1920; **53**, 150, 1921; *Physical Review*, **12**, 168, 1918. *Science*, **19**, 138, 1919.

2. EXPERIMENTAL METHOD

The vacuum sparks were produced in a glass bulb about 12 cm in diameter provided with a quartz window, which in the case of the more easily vaporized metals was removed to a distance of 30 cm on account of the sputtering. The iron electrodes consisted of cylinders about one cm long and one cm in diameter, tapering to blunt points. These cylinders were screwed on the ends of aluminum rods, which projected to the outside through side tubes and were sealed in with sealing wax. The iron electrode was retained as cathode throughout the experiments, while anodes of the other metals were made by wedging a piece of the metal into a hole in the end of the aluminum rod. The distance between the electrodes was about 1.5 mm in the case of iron, but it amounted to 3 or 4 mm with some of the other metals, which were largely vaporized during the exposure.

Evacuation was effected with a Langmuir diffusion pump, the mercury vapor being kept away from the discharge by a trap immersed in solid CO_2 and ether.

The discharge was produced with a large induction coil used with an electrolytic interrupter. A current of from 15 to 20 amperes was passed through the primary. To secure the necessary capacity across the secondary, four parallel plate condensers, consisting of copper plates about 56 by 61 cm separated by 5-mm glass plates, were connected in series. The whole was immersed in oil to prevent leakage.

About eight hours were required to obtain a spectrogram of iron, because of the tendency of the discharge to pass over into an ordinary gas discharge, with the liberation of gas from the electrodes, and because it was necessary to keep the sealing wax joints cool. The total duration of the spark discharge was estimated at about thirty-five minutes for iron and not more than ten minutes in the case of cadmium. The iron required a much higher potential gradient between the electrodes for the production of the spark than the other metals.

A spectrograph with concave grating of 20-cm radius and ordinary portrait films were used in obtaining most of the spectrograms. The iron spectrum was photographed in the region 4000–6600 A by using an aesculin filter and films sensitized with a Wallace three-dye solution. Fairly rich spectra of iron and titanium were obtained also with a one-meter concave grating spectrograph.

The vacuum-spark spectra of magnesium and cadmium with comparison arc and spark spectra are reproduced in Plates IV and V.

3. DISCUSSION OF RESULTS

The vacuum-spark spectra of calcium and magnesium are practically identical with the spark spectra in air.

The cadmium vacuum-spark spectrum retains arc lines in somewhat greater relative intensity than they possess in the ordinary spark. The line λ 3261 is an example. This line came out also with marked intensity in the luminescence spectrum produced by cathode-ray bombardment.[1]

Titanium shows an even greater tendency to retain the arc lines. The spectrum is plainly an enhanced one, however, as is evidenced by the brightness of the lines λλ 4164 and 4172, for example, the ratios of whose intensities in the spark and the arc, as given by Exner and Haschek, are 20:2 and 15:1, respectively.

The vacuum-spark spectrum of iron, on the other hand, seems to resemble more closely the arc spectrum. The low-temperature lines found in the arc are missing, and the lines which are stronger at the poles and in the core of the arc are relatively more intense in this source. Even in this spectrum a close examination reveals the presence of the enhanced lines, such as λλ 4508, 4515, 5018, and 5316. Hα comes out very strongly and sharply in the photographs of the iron spectrum made in this region.

The varying tendency exhibited by these different metals toward the production of enhanced lines suggests a resemblance of the vacuum-spark spectra to the luminescence spectra produced by cathode-ray bombardment in a high vacuum. It is probable that the conditions of emission are very similar in the two sources. The difficulties in the way of obtaining rich spectra are considerably less in the case of the vacuum-spark.

MOUNT WILSON OBSERVATORY
September 1921

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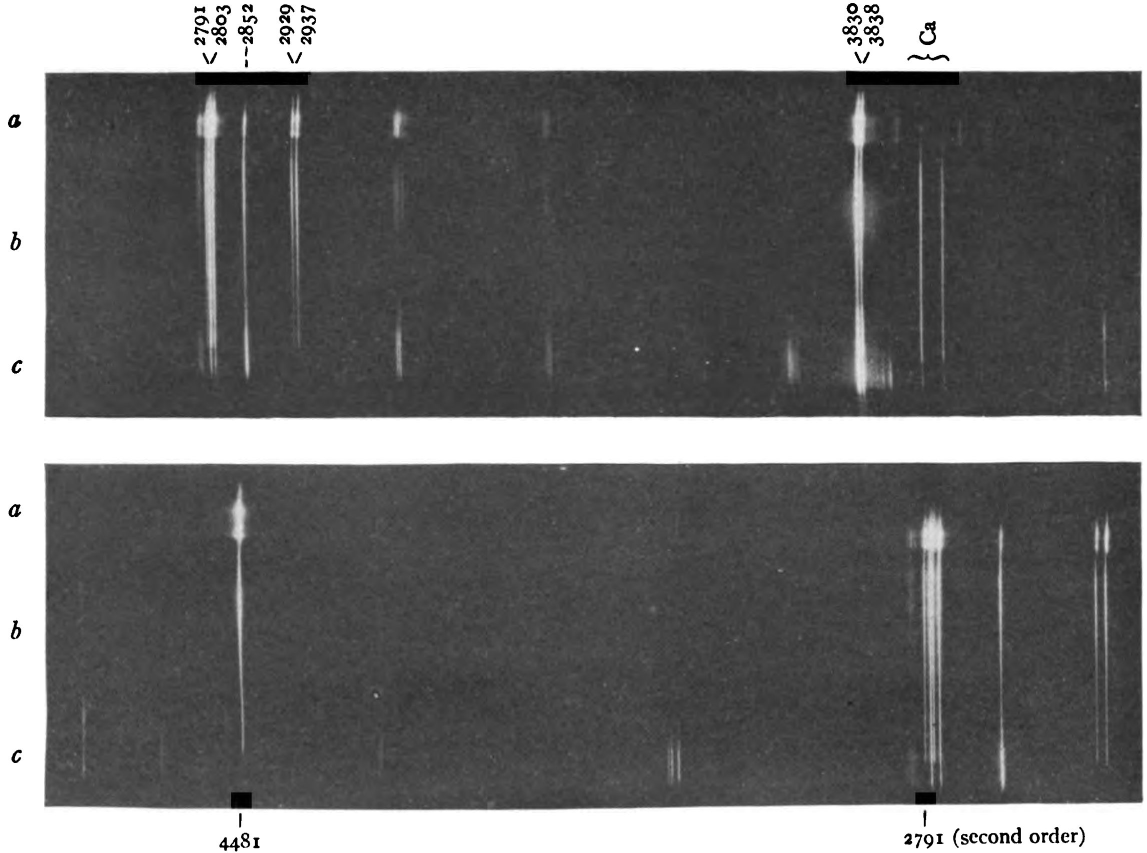
[1] *Astrophysical Journal*, **44**, 303, 1916; **49**, 224, 1919.

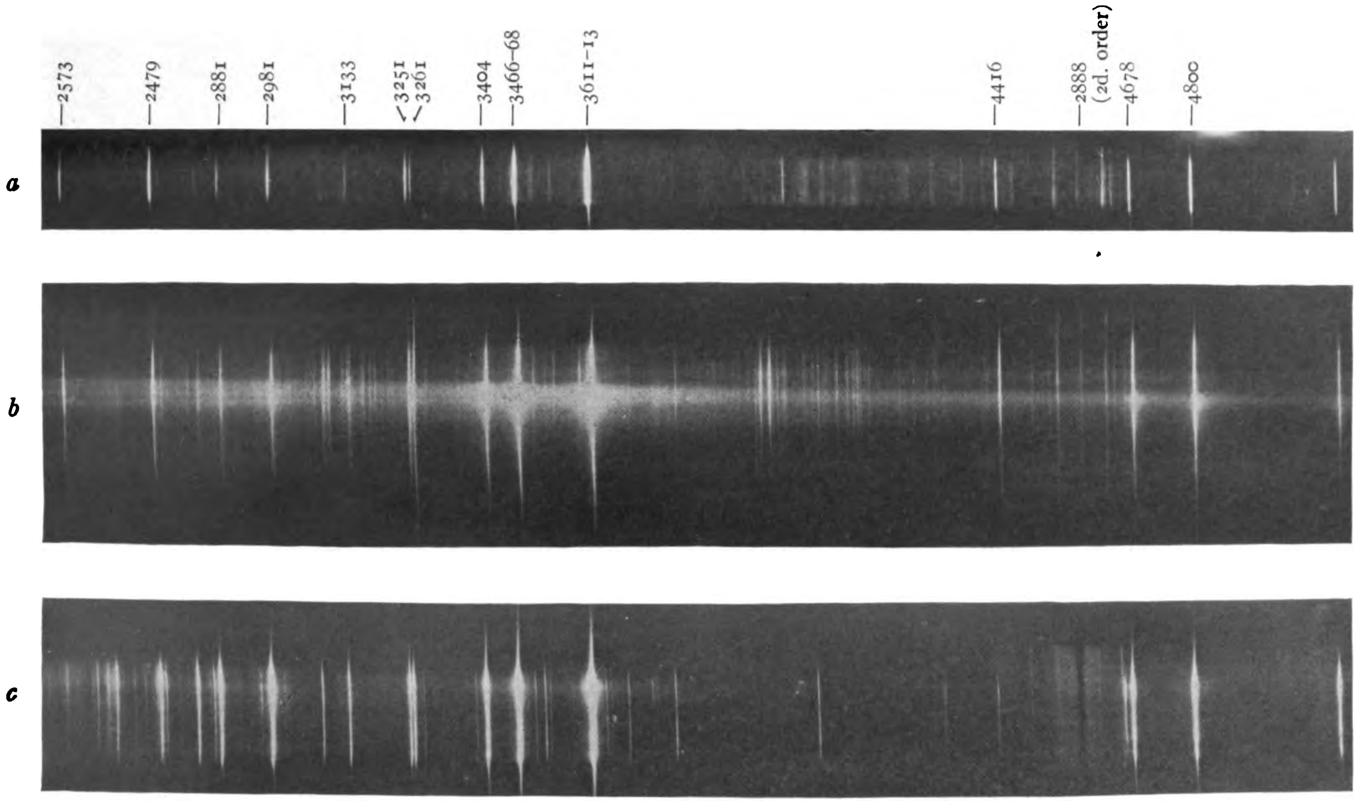
PLATE IV



SPECTRUM OF MAGNESIUM

a Spark in air
b Vacuum-spark
c Arc

PLATE V



Spectrum of Cadmium

a Spark in air
b Vacuum-spark
c Arc

THE MASSES AND DENSITIES OF THE STARS[1]

BY FREDERICK H. SEARES

ABSTRACT

Mean magnitudes and masses of visual binaries and dwarf stars of the different spectral types.—A statistical comparison of the absolute magnitudes (M_b) of 105 visual binaries, the hypothetical magnitudes (M_c) of these and 400 other visual binaries (computed by Jackson and Furner, assuming the combined mass, $\mu=\mu_1+\mu_2$, of each binary to be 2), and the absolute magnitudes (M_s) of 1152 single stars (including some binaries) by Kapteyn and Adams, gives the mean difference $(\overline{M}_b-\overline{M}_c)=(\overline{M}_s-\overline{M}_c)-0.3\pm0.1$ for each type; that is, $\overline{M}_b=\overline{M}_s-0.3\pm0.1$. This indicates that the selection used is sufficiently homogeneous, also that the variation of the mean magnitude of binary star components with spectral type is the same as that of single stars. The values of $\overline{M}_s$ (Table IV and Fig. 1) for dwarf stars of types Bo, Ao, Fo, Go, Ko, and Ma are -1.60, $+0.70$, 2.40, 4.35, 5.90, and 9.80 respectively. The geometric mean mass for each spectral type of visual binary was next computed from the formula $\log\mu=\log 2-0.6\ (\overline{M}_s-0.3-\overline{M}_c)$. The values of μ, slightly corrected for the effect of selection as determined by a study of the binaries of known parallax, give, when multiplied by ½ to reduce to single stars, a first approximation to the geometric mean masses of single stars along the dwarf branch of Russell's diagram, the values for types Bo, Ao, Fo, Go, Ko, and Ma, coming out 10, 6.0, 2.5, 1.0, 0.68, and 0.59, respectively, in terms of the sun's mass (Table IV). The data for visual binaries of known parallax indicate that the distribution of $\log\mu$ is approximately Gaussian and that the *dispersion in mass* is small, probably half the stars of each type having masses between 0.7 and 1.5 times the mean (Table VIII).

Mean velocity and energy of dwarf stars of different spectral types.—The mean square space-velocity corresponding to the mean magnitude was determined from the available data for each type, and these values were multiplied by the corresponding mean masses. The mean energy for each type so obtained is practically the same for all types from Ao to Ma, in spite of a 10 to 1 variation in mass (Table IX). The logarithm of the *mean kinetic energy constant* is 3.27 ± 0.036 (average deviation per type) in terms of sun's mass and km/sec., corresponding to 3.5×10^{46} ergs. The mean radial energy is also approximately constant for all except the B-type stars, which seem to be abnormal in this as in other respects. Whatever the cause may be, the distribution of energy among these stars seems to approximate equipartition.

Masses of stars as functions of magnitude and spectral type were then computed from the values of the mean velocity for each magnitude and type, assuming the mean energy independent of magnitude as well as of type (Table X). In Figure 2, curves are given showing M as a function of type for various constant values of the mass. To get more reliable data for the giants, the masses of 28 Cepheids were computed from the period, magnitude, and surface brightness of each, using a formula based on the pulsation theory (Table XVI), and Figure 2 was revised to agree with these values, giving Figure 3. If the masses computed in accordance with Eddington's theory of Cepheids (Table XIX) are adopted, a further slight revision will be necessary. The constant mass lines would be less irregular if bolometric magnitudes were used instead of visual. For the Cepheids, the probable *dispersion in mass* for a given M and type is only about 20 per cent. This result taken with the data for visual binaries indicates that in general the luminosity of a star is very closely determined by its spectral type (temperature) and its mass.

Effective temperature, color index, and surface brightness of stars of different spectral types.—The effective or black-body temperatures of the giants as determined from

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[1] *Contributions from the Mount Wilson Observatory*, No. 226.

Wilsing's spectrophotometric measures of c/T for 199 stars, vary from 2890° for Mc to 10,500° for Bo, while those of the dwarfs as determined from Mount Wilson color indices (provisional values, hitherto unpublished, also given for the giants) reduced to the same system, vary from 3330° for Ma to 6080° for F5 (Table XII). The difference between dwarfs and giants of the same type is too great to be neglected, the later-type dwarfs being considerably hotter. The surface brightness j as a function of temperature may be computed (1) from the difference between visual and bolometric magnitudes as determined by Eddington ($j=M-M_B-10 \log T+\text{const.}$) and (2) from Hertzsprung's formula derived from Planck's radiation law ($j=2.3\ (c/T)^{0.93}+\text{const.}$), both formulae giving practically the same results (Table XI). Combining these values with the temperatures of the stars of various types, we get the surface brightness as a function of spectral type, the results varying from 4.81 for $T=2890°$ to -2.28 for $T=10{,}500°$, expressed in magnitudes, sun$=0$ (Table XII). As a check on these values, the surface brightness of three stars, α Orionis, α Boötis, and α Scorpii, whose diameters have been determined directly by Pease, were computed and found to agree with the interpolated values within $\pm 0^{M}.2$ in each case (Table XIII).

Diameter and density of stars as a function of magnitude and spectral type were readily computed from the surface brightness, absolute magnitude, and mass (Table XIV). For dwarfs the diameter decreases from 6.8 to 0.54 (sun$=1$) and the density increases from 0.045 to 5.4 (water$=1$) as the type changes from Bo to Ma, while for giants of zero magnitude the corresponding changes of diameter and density are from 3.2 to 66 and from about 0.3 to 10^{-5}. When M is plotted as a function of type for various constant values of the density (Fig. 2), the equidensity lines are straight and nearly parallel and quite regularly spaced, so that the equation: $\log \rho=-1.22\,S+0.57\,M+1.11$, where S is the type number beginning with 0 for Bo, represents the results quite closely for a wide range of density. For dwarfs of types B to F, the values agree satisfactorily with the values found by Shapley for the densities of eclipsing variables of these types. The giant part of the diagram, however, is less reliable, and is later modified in Figure 3 to agree with the values for 28 Cepheids (Table XVI) computed from their periods by the use of the theoretical formula: $\log \rho=-2 \log P+\text{const.}$ Adoption of Eddington's formula for Cepheids would lead to only a slight further modification (see Table XVII).

Relation of the observed variation of mean mass and of M with spectral type to the problem of stellar evolution.—For the dwarf branch, the decrease in mass from 10 for Bo to 0.6 for Ma may be partly, and perhaps wholly, accounted for by selection, the earlier and brighter stars being observed throughout a larger volume; further, only massive stars can attain high temperatures. The possibility that *mass may decrease with loss of energy by radiation* is also suggested, a possibility which agrees with the theory of relativity and does not necessarily conflict with Newtonian mechanics. For the giants, the observed change in M with spectral type (mass constant) is difficult to account for. Probable changes in mean atomic weight are insufficient. Changes in the opacity are doubtless involved, and likely also the source of energy, which almost certainly is not wholly gravitational and probably not independent of the time; for instance, the probable degree of ionization involves an amount of energy greater than seems to be available from gravitational forces. The intersection of equal-mass lines with the dwarf branch at large angles (Figs. 2, 3), however, does not conflict with the gravitational contraction hypothesis.

Eddington's radiative equilibrium theory of giants.—Although this involves the assumption that the ratio of the energy flow to the gravitational acceleration is constant throughout a star, whereas the ratio probably decreases considerably from center to outside surface, the calculated variation of magnitude with mass for a given spectral type is in fair agreement with observed values (Table XXII), but for a given mass the absolute magnitude is not constant as required by theory. *For Cepheids*, the theory gives the formula: $\log \rho=-2 \log P-2 \log (\gamma a)^{\frac{1}{2}}+\text{const.}$, and this combined with the empirical relation, mass is proportional to radius, which Eddington's results suggest as a *condition for Cepheid variation*, gives: $M=-5 \log P+j-5 \log (\gamma a)^{\frac{1}{2}}+\text{const.}$, which agrees with the observed variation of M with P for $P>2$ days, quite closely (Table XVIII and Fig. 5). This shows the consistency of Eddington's theory of Cepheids with his theory of radiative equilibrium

and tends to confirm the relation of mass to radius, which is equivalent to the condition that the average heat content per unit mass is constant. Various consequences of this condition are given. The theoretical values of mass and density given by the pulsation theory are close to those indicated by Figure 3 (Table XVII).

Ionization and mean atomic mass inside stars.—Theoretical computations indicate that the temperature and pressure at various points inside a star change in such a way as to keep the degree of ionization of the same order throughout (Table XX); hence the mean atomic mass is also about constant. When we consider giant stars of constant mass but of increasing temperature, we find that while the densities change very greatly, the high degree of ionization in the earliest stages is maintained with but little change (Table XXI), so that the mean atomic mass should decrease but slowly as development proceeds.

Comparison of spectroscopic parallaxes of visual binaries with the hypothetical parallaxes of Jackson and Turner shows systematic differences for their Table II (Table III).

Information on the masses of stars has its origin in what can be learned of the masses of binary systems, spectroscopic and visual. Any discussion of spectroscopic binaries from the standpoint of mass is complicated by lack of definite information on the inclination of orbital planes, while in the case of visual binaries the lack is that of accurate knowledge of stellar distances. Approached from either standpoint, the problem, with rare exceptions, must be treated by statistical methods whose application presupposes an abundance of data. Any solution is therefore involved in some uncertainty, which is not lessened by the fact that a transfer of results thus obtained to the stars at large involves assumptions of comparability: we do not know, for example, how the mass of an average binary is related to that of an average single star of the same type and luminosity, or even that the relation is the same for all classes of stars. The uncertainty, however, is probably not great; at any rate, we can proceed only by making such assumptions.

The following discussion is based on the visual binaries. These lead to values of the masses of stars along the dwarf branch of the Russell diagram, including those of B-type spectra. The average mass decreases continuously with advancing spectral type. The combination of these results with the space velocities of large numbers of stars recently published by Adams, Strömberg, and Joy shows that the mean kinetic energy of translation of the A's and the dwarfs is approximately constant. The principle of equipartition is then used to derive values of the masses of stars of other luminosities. The distribution curves thus obtained for mass as a function of spectral type and absolute magnitude are a first approximation in so far as they concern stars of highest luminosity, but in the vicinity of the dwarf branch they should be reliable within a moderate percentage.

Valuable information on density has at various times been obtained from the eclipsing variables, but it has been difficult to correlate these results with other stellar characteristics, except loosely, because here again the distances, and hence the luminosities, of individual stars are inadequately known. Density, however, depends in a simple way on mass, total luminosity, and surface brightness. The surface brightness can be computed from radiation formulae, the results being checked in part by their agreement with Pease's measures of angular diameters. Hence with mass once known as a function of spectral type and absolute magnitude, the values of the density follow as a matter of course. The results thus found for mass and density are subject to control and readjustment with the aid of the Cepheid variables, whose relative densities and masses can be calculated directly from their periods, spectra, and surface brightness.

The lines of equal mass for the giant stars in Russell's diagram are irregular curves, considerably inclined to the axis of zero absolute magnitude. If therefore a giant star in its evolutionary development follows an equal-mass line, its luminosity does not remain constant during the giant stage as has often been supposed. The curves of equal density are approximately linear and parallel to the line of maximum frequency for the dwarfs. The correlation of mass with spectral type along the dwarf branch can be explained, in part at least, as a consequence of selection, although other factors may also enter. The concluding sections of the paper discuss the relation of these results to certain phases of Eddington's theory.

I. THE METHOD

The mean masses of the visual binaries of different spectral types may be found by a statistical comparison of the absolute magnitudes corresponding to their hypothetical parallaxes with the absolute magnitudes of stars of known parallax. The relation

$$\mu=\mu_1+\mu_2=\frac{a^3}{\pi^3 P^2} \qquad (1)$$

leads at once to

$$\log \mu = k - 0.6M \tag{2}$$

where

$$k = 3 \log a - 2 \log P + 0.6m + 3$$

and where the absolute magnitude is defined by

$$M = m + 5 + 5 \log \pi. \tag{3}$$

For a hundred systems of known orbital elements Jackson and Furner[1] have used equation (1) to calculate the hypothetical parallax π_c corresponding to an assumed mass $\mu = 2$. For about 450 systems of unknown elements they have calculated hypothetical parallaxes from a formula depending on changes in position angle and distance, the assumption for the mass again being $\mu = 2$. The absolute magnitudes M_c corresponding to the hypothetical parallaxes for the two groups of stars are given in Tables I and II of their paper.

For any system of known elements, by (2),

$$0.30 = k - 0.6\,M_c$$

and by combination with (2) itself,

$$\log \mu = 0.30 - 0.6\,\Delta M \tag{4}$$

where $\Delta M = M - M_c$. Hence for a group of stars[2]

$$\left.\begin{aligned} \overline{\log \mu} = \log \underline{\mu} &= 0.30 - 0.6\,\overline{\Delta M} \\ \overline{\Delta M} &= \overline{M} - \overline{M}_c \end{aligned}\right\} \tag{5}$$

in which $\underline{\mu}$ represents the geometrical mean mass.

Equations (4) and (5) are the basis of the present discussion. Since the hypothetical parallaxes derived from changes in position angle and distance involve the mean inclination of the orbit-planes, the individual values of π_c thus found will differ from those obtained with the rigorous relation (1). Equation (4) therefore holds for individual stars only in the case of systems of known elements. Formulae (5), however, are valid for any representative group of stars, whether their elements are known or not, and can therefore be applied to objects in both the lists of Jackson and Furner.

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[1] *Monthly Notices*, **81**, 2, 1920.

[2] Upon re-reading Russell's address, "Relations between the Spectra and Other Characteristics of the Stars," *Publications of the American Astronomical Society*, **3**, 22; *Popular Astronomy*, **22**, 275, 331, 1914, I find that in principle the method is essentially one used by him.

To determine the mass from equations (4) and (5), the values of M, or at least of $\overline{M}$, must be known for the various spectral types. For a certain number of systems direct determinations of distance are available. Hence, M is known, and the use of (4) gives at once values of μ for the individual stars. These results will be affected by large uncertainties, but the means for groups of stars should be accurate within a small percentage. For most of the binaries, however, only the hypothetical absolute magnitude M_c is known; no other approximation for M is available, and we must proceed indirectly. This we might do by using values of $\overline{M}$ derived from single stars of known parallax whose selection with respect to the stars as a whole presents the same characteristics as those of the binaries themselves. Similarity of selection in the two lists of stars—binaries and single stars—is, however, difficult to attain in practice. It will be sufficient if we can find a list of single stars whose differences in selection as compared with the binaries are the same for all spectral types.

Let $\overline{M}_s$ represent the mean absolute magnitude for a given spectral type of the single stars of known parallax, and let $\overline{M}$ be the corresponding magnitude of the binaries of the same type. We may then write

$$\overline{M}=\overline{M}_s+\delta M \tag{6}$$

where δM indicates the influence of difference in selection. If this difference is the same for all types, δM will be constant, and we shall then speak of the selection as homogeneous.

The second of (5) then becomes

$$\overline{\Delta M}=\overline{M}_s-\overline{M}_c+\delta M=\overline{\Delta M}_s+\delta M \tag{7}$$

where $\overline{M}_c$ is the mean hypothetical absolute magnitude of the binaries of the spectrum under consideration. The value of $\overline{\Delta M}_s=\overline{M}_s-\overline{M}_c$ is found by comparing the two lists of stars for which it is presupposed that the selection for different spectral types is homogeneous, while the constant δM is given by

$$\delta M=\overline{\Delta M}-\overline{\Delta M}_s=\overline{M}-\overline{M}_c-\overline{\Delta M}_s \tag{8}$$

applied to the systems of independently determined parallax referred to above. M represents here the magnitude corresponding to the measured parallax of a binary whose hypothetical absolute magnitude is M_c. In other words, we derive δM by (8) from the hundred or more binaries of measured parallax, calculate $\overline{\Delta M}_s$ for each type by comparing the five hundred and fifty-odd stars in the lists of Jackson and Furner with the homogeneously selected single stars of known parallax, form $\overline{\Delta M}$ by (7), and, finally, solve for μ by means of the first of equations (5). Since the question of homogeneity of selection can be put to a direct test, the only assumption underlying this procedure is that the relation of mass to luminosity is the same for both binaries and single stars, which may be accepted as plausible at least.

The values of $\overline{M}_s$ and $\overline{M}_c$, whose difference is $\overline{\Delta M}_s$, are readily found by a graphical process. Plotting the values of M_c against spectral type, we find for the binaries a frequency diagram similar to that first given by Russell. The late-type giants are not numerous, but the dwarf branch is well defined. The points of maximum frequency for the dwarfs define a curve which joins smoothly with the line of modal values of the B- and A-type stars. This curve expresses the variation of $\overline{M}_c$ with spectral type, the giants of late type being disregarded altogether for the present. For $\overline{M}_s$ a similar curve was found from the data in the list of spectroscopic parallaxes,[1] supplemented by those for about 430 helium stars whose parallaxes have been determined by Kapteyn.[2] These two curves are shown in the central part of Figure 1. Their ordinates and differences in ordinates are given in the second, third, and fourth columns of Table IV. The data for the $\overline{M}_s$ curve could have been greatly increased by including results from trigonometric parallaxes. Little would have been gained in precision, however, for, as it is, the number of stars entering into the $\overline{M}_s$ curve is more than twice that for the $\overline{M}_c$ curve.

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[1] Adams, Joy, Strömberg, and Burwell, *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

[2] *Mt. Wilson Contr.*, Nos. 82, 147; *Astrophysical Journal*, **40**, 43, 1914; **47**, 146, 255, 1918.

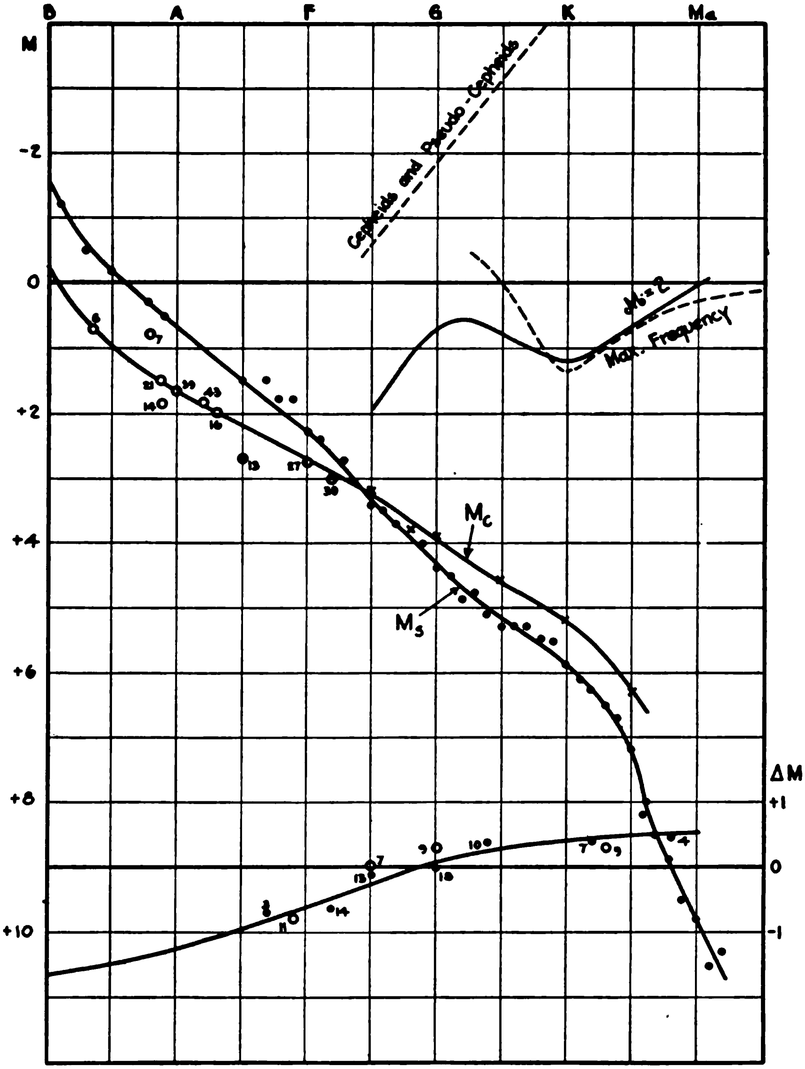

FIG. 1.—The curve M_s is the line of maximum frequency of M for the B stars of Kapteyn and the dwarfs from the list of spectroscopic parallaxes. M_c is the similar curve for the hypothetical absolute magnitudes of Jackson and Furner. The differences in the ordinates, corrected for zero point, are shown by the curve below (ΔM). The remaining curves refer to the discussion in section 16.

If we may assume that the selection for different spectral types is homogeneous, the differences in the ordinates of the two curves represent the values $\overline{\Delta M_s}$ as a function of spectrum. Irregularities of selection, or a progressive change in selection with type, will produce deformations or a tilt of the curve for $\overline{\Delta M_s}$. The curve for $\overline{\Delta M}$, formed directly from the curves for $\overline{M}_c$ and $\overline{M}_s$ and corrected for δM, is shown in the lower part of Figure 1 (the points are explained later). Since the curve is smooth, we conclude that there are no marked irregularities of selection, or at least that their influence has been minimized in drawing the curves for M_s and M_c.

2. INFLUENCE OF SELECTION ON THE STATISTICAL COMPARISON

To examine the question of selection more closely, consider the numbers of binaries and single stars of known parallax in each interval of apparent magnitude. Excluding the late-type giants and a few scattering stars at the ends of the sequence of apparent magnitudes, we find the data in the first two divisions of Table I. If the selection for the two lists were the same (not merely homogeneous), the ratios of the corresponding numbers for each type would not vary with the magnitude. Difference in selection for the binaries and single stars of any type implies a change in ratio with increasing magnitude; and, if the selection is homogeneous, the ratios for different types will show the same change. Absolute values of the ratios are of no significance, and for ease of comparison the ratios for each type have been multiplied by a constant factor which reduces that for median magnitude 6.5 to unity. Further, the numbers for the G5–K5 binaries have been combined with the mean numbers for single stars of F0–F9 and K0–K9 types.

The results in the third division of Table I indicate a satisfactory degree of homogeneity. The irregularities affecting the bright G5–K5 stars are unimportant, since the individual groups include only two or three stars each. The only deviations requiring comment are those shown by the faint G5–K5 stars and the faint B's. The latter arise from the fact that Kapteyn's lists of B stars include none fainter than the 6th magnitude. The binaries involved are ten in number, all of the 7th magnitude or fainter, and all but one or two of types B8 or B9. Any error in the corresponding $\overline{\Delta M}_s$ must be small, for there is no pronounced irregularity in the curve. The value of $\overline{M}_s$ for Kapteyn's earlier B's

TABLE I

NUMBERS OF STARS—INFLUENCE OF SELECTION

Spectrum	Median Apparent Magnitude								Total No.
	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	
	Binaries—Jackson and Furner								
Oe5–B9	1	1	7	7	7	8	2	0	33
A0 –A9	4	3	12	32	42	25	15	1	134
F0 –G0($M>+1$)	0	6	15	36	66	78	54	2	257
G5 –K5($M>+3$)	0	1	3	4	9	24	39	1	81
Total	......	......	......	......	......	......	......	......	505
	Single Stars*—Kapteyn and Adams								
Oe5–B9	23	42	137	194	30	0	0	0	426
A0 –A9	0	7	11	11	8	3	(2)	(1)	43
F0 –F9($M>+1$)	2	14	48	105	63	50	29	3	314
G0 –G9($M>+3$)	0	5	10	35	57	54	40	8	209
K0 –K9($M>+4$)	0	1	2	14	26	43	62	12	160
Total	......	......	......	......	......	......	......	......	1,152
	Ratio—Binaries/Single Stars								
B	......	0.1	0.2	0.2	1.0	∞	∞	......	......
A	......	0.1	0.2	0.6	1.0	1.6	1.5	0.2	......
F	......	0.4	0.3	0.3	1.0	1.5	1.8	0.7	......
G5–K5	......	1	3	0.8	1.0	2.2	3.5	0.5	......
	Ratio—Jackson and Furner/Pickering								
B	......	......	7	2	1.0	0.52	0.08	0.00	......
A	......	......	3	2	1.0	0.29	0.04	0.00	......
F	......	......	3	2	1.0	0.38	0.10	0.01	......
G5–K5	......	......	3	2	1.0	0.80	0.54	0.02	......
	Ratio—Kapteyn and Adams/Pickering								
B	......	......	34	13	1.0	0.00	0.00	0.00	......
A	......	......	17	4	1.0	0.18	0.03	0.02	......
F	......	......	12	6	1.0	0.25	0.05	0.01	......
G	......	......	3	4	1.0	0.33	0.08	0.02	......
K	......	......	0.4	1.5	1.0	0.46	0.34	0.12	......

* The "single stars" include the brighter members of a few binaries.

can scarcely have been affected by the limitation in apparent magnitude, for these stars are all so luminous that the limitation is of little consequence.

Among the stars fainter than $m=7.0$ there is an excess of G5–K5 binaries which represents a considerable fraction of the total number of G5–K5 stars. The corresponding $\overline{M}_c$ may be slightly too large. But, if appreciable, this effect must also be small, for as the stars actually observed in the intervals of fainter apparent magnitude are successively added, the value of $\overline{M}_c$ increases very slowly. A direct test is afforded by the binaries of measured parallax considered in a later paragraph.

The number of A stars in the spectroscopic list is small, but this does not seriously interfere with the determination of the curve of modal values because there are well-determined points for the early B's and for the F's. It does suggest, however, the advisability of an independent test of the selection of the much larger group of A's among the binaries, whose numbers can be considerably changed without greatly affecting the ratios in the middle section of Table I. To cover this point and as a further general test, both groups of stars have been compared with the counts of W. H. Pickering.[1] Using means of his values for the galactic poles and the galaxy itself (except for the B's, for which the galactic counts alone have been employed), we find the ratios in the last two sections of Table I. There is here some progression with type due to the fact that Pickering's counts include the late-type giants; but the ratios for the A-type binaries are in general agreement with those of the adjacent classes, which answers the question as to the selection of the A's. The large ratios for the bright B's in the last section of the table show merely what appears directly from the counts, namely, that Kapteyn's lists are not complete beyond the 6th magnitude. There seems also to be a defect of fainter A's in the spectroscopic list, a conclusion apparently confirmed by the fact that the points for these stars lie above the smooth $\overline{M}_s$ curve shown in Figure 1. It seems clear that none of the questions raised is serious and that we may accept the values of $\overline{\Delta M_s}$ found in the manner described and proceed to the determination of δM.

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[1] *Publications of the Astronomical Society of the Pacific*, 33, 140, 1921.

3. ZERO POINT OF THE CURVE FOR MASS AND SPECTRUM

It will be noted that the values of $\overline{\Delta M_s}$ determine the rate of change of mass with spectral type, while the constant, δM, determines the zero point. For the calculation of δM only the spectroscopic parallaxes have been used, since these are based on a homogeneous system and are sufficiently numerous to afford an excellent determination. In forming values of $M-M_c$ for individual stars to be substituted into (8), two points must be borne in mind: First, the values of M given in the spectroscopic list correspond to the most probable values of the parallaxes, and, because of the logarithmic relation connecting absolute magnitude and parallax, are not the most probable magnitudes themselves. To obtain the latter, a small constant correction of +0.08 mag. must be applied to the tabular values.[1] Second, the values of M_c given by Jackson and Furner usually represent the total luminosity of the system. To make them comparable with values of M in the spectroscopic list, they must be reduced to the brighter component.

A summary of the results for δM is given in Table II, the two lists of Jackson and Furner being treated separately. The values of $\overline{\Delta M_s}$ required for equation (8) were interpolated from the fourth column of Table IV. The adopted result is $\delta M = -0.3$, whence equation (7) becomes

$$\overline{\Delta M} = \overline{\Delta M_s} - 0.3 . \tag{9}$$

It will be noted that there is no marked progression in the values of δM with type, which confirms, in part at least, the foregoing conclusion as to homogeneity of selection. This is also shown by the lower part of Figure 1, in which the curve is that representing equation (9), while the points correspond to the values of $\overline{M-M_c}$ in Table II. With the exception of the constant term, the curve depends wholly upon the statistical comparison, while the points are derived from values of M and M_c referring to the same star and are uninfluenced by any possible dissimilarity in selection. The agreement of the points with the curve is, however, within the uncertainty affecting their positions.

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[1] *Mt. Wilson Contr.*, No. 199, p. 15; *Astrophysical Journal*, 53, 27, 1921.

TABLE II

ZERO POINT OF CURVE FOR $\overline{\Delta M}_s$

J AND F, TABLE I					J AND F, TABLE II				
Sp.	$\overline{M-M}_c$	$\Delta\overline{M}_s$	δM	No. Stars	Sp.	$\overline{M-M}_c$	$\Delta\overline{M}_s$	δM	No. Stars
A7.....	−0.7	−0.5	−0.2	3	A9....	−0.8	−0.4	−0.4	11
F2.....	−0.6	−0.2	0.4	14	F5....	0.0	0.0	0.0	7
F5.....	−0.1	0.0	0.1	13	G0....	+0.3	+0.4	−0.1	9
G0....	0.0	+0.4	0.4	18	K3....	+0.3	+0.8	−0.5	9
G4....	+0.4	+0.6	0.2	10					
K2....	+0.4	+0.7	0.3	7					
K8....	+0.5	+0.8	−0.3	4					
Mean and total......			−0.29	69	Mean and total......			−0.27	36

A further detail bearing on the question of selection is the circumstance that the constant δM arises almost wholly from the use of the tabular values of M_c for the derivation of the curve for $\overline{M}_c$. Since these generally represent total luminosity, the reduction to the brighter component appears in δM, and in the mean is nearly equal to the value found for this constant. The effect upon $\overline{\Delta M}_s$ of the obvious differences in selection shown in Table I is therefore very small. Consequently, the influence of small deviations from homogeneity in selection in passing from type to type may be regarded as altogether negligible.

4. SYSTEMATIC DIFFERENCES IN THE HYPOTHETICAL PARALLAXES

Although the two mean values of δM given in Table II are practically identical, this was not the case when the calculation was originally made, the difference then amounting to several tenths of a magnitude. This discordant result has been traced to a systematic difference between the parallaxes of the two lists of Jackson and Furner, the character of which may be seen from the comparisons shown in Table III. For systems of known elements (J and F, Table I) the agreement of the mean hypothetical parallaxes with the spectroscopic results is excellent; the one large difference +0″014 is only 7 per cent of the corresponding π_c and would be reduced to +0″004 by the rejection of a single star, Bu 2109.

TABLE III

COMPARISON OF SPECTROSCOPIC AND HYPOTHETICAL PARALLAXES

π_c	J AND F, TABLE I			J AND F, TABLE II			J AND F, TABLE I—TABLE II		
	Mean π_c	$\pi-\pi_c$	No.	Mean π_c	$\pi-\pi_c$	No.	Mean π_c*	$\pi-\pi_c$	No.
<0″020	0″016	+0″001	7	0″015	−0″002	8	0″015	−0″001	6
0.020–0.049........	0.030	−0.001	28	0.036	−0.009	16	0.024	−0.007	12
0.050–0.099........	0.065	−0.003	22	0.064	−0.012	13	0.043	−0.012	8
>0.100	0.198	+0.014	10	0.209	−0.071	1	......	....	....

* The intervals for π_c are $\gtreqless$0″020, 0″021–0″029, and $\lesseqgtr$0″030.

For the stars in the second list, however, the agreement for the first group alone is within the uncertainty; for the others the differences are systematic and an important percentage of the parallaxes themselves. Further, the same difference is shown by a direct comparison of the hypothetical parallaxes of twenty-six stars which occur in both the lists of Jackson and Furner. The means, arranged in three groups, are in the last division of Table III. For half of these stars no spectroscopic parallaxes are available, and to that extent the second comparison is independent of the first.

The difference, which is about 20 per cent of the parallax, corresponds to a difference of 0.4 mag. in the two values of δM and fully accounts for the discordance. For the final determination of δM, the hypothetical parallaxes from the second list were reduced to the spectroscopic system in accordance with the indications of Table III, in order that all the results may refer consistently to this system. The resulting value of δM is that given in the right-hand division of Table II.[1]

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[1] I am greatly indebted to Mr. Adams for the spectroscopic parallaxes of nineteen binaries in the first list of Jackson and Furner (six stars also occur in the second list) which have been obtained since the reductions described in the text were finished. These reductions have not been revised, but I have satisfied myself that the inclusion of the new data would not change the results in section 4 or the following sections

5. PROVISIONAL MASSES—EFFECT OF SELECTION

The derivation of values of the masses now requires only the substitution of $\overline{\Delta M}$ from (9) into the first of (5). The results for log $\underline{\mu}$ are in the fifth column of Table IV. The first and last values depend upon extrapolations, and it is possible that those for B5 and A0 are also affected by considerable uncertainty. So far as the statistical comparison is concerned, the remaining values should be dependable within 5 or 10 per cent.

TABLE IV

ADOPTED GEOMETRICAL MEAN MASS ($\odot = 1$), SPECTRAL TYPE AND ABSOLUTE MAGNITUDE

Sp.	$\overline{M}_s$	$\overline{M}_c$	$\overline{\Delta M}_s$	Provisional Log $\underline{\mu}$	Corr. for Selection	Vis. Binaries		Single Stars	
						Log $\underline{\mu}$	$\underline{\mu}$	Log $\mathcal{M}$	$\mathcal{M}$
B0	−1.60	(−0.25)	(−1.35)	1.29	−0.04	1.25	18	1.01	10
B5	−0.20	+1.00	1.20	1.20	− 4	1.16	14	0.92	8.3
A0	+0.70	1.65	0.95	1.05	− 3	1.02	10.5	0.78	6.0
A5	1.50	2.15	0.65	0.87	− 3	0.84	6.9	0.60	4.0
F0	2.40	2.70	−0.30	0.66	− 2	0.64	4.4	0.40	2.5
F5	3.32	3.27	+0.05	0.45	− 2	0.43	2.7	0.19	1.5
G0	4.35	3.95	0.40	0.24	− 1	0.23	1.7	9.99	1.0
G5	5.20	4.60	0.60	0.12	0	0.12	1.3	9.88	0.76
K0	5.90	5.20	0.70	0.06	+ 1	0.07	1.2	9.83	0.68
K5	7.10	6.30	0.80	0.00	+ 3	0.03	1.1	9.79	0.62
Ma	+9.80	(+8.95)	(+0.85)	9.97	+0.04	0.01	1.0	9.77	0.59

These results can be accepted, however, only with some reservation, for it is easily seen that the visual binaries are a selected class of stars in the characteristic of mass itself, and especially is this true of the more distant objects. Since motion is detected only with difficulty, or not at all, in case the period is very long, equation (1) shows that the selection of large masses is favored to an increasing degree as more and more distant stars are considered. For a given value of a the factor a^3/π^3 increases rapidly with increasing distance; the smaller masses are therefore excluded in succession because they correspond to periods so long that motion has not been observed. For large values of π a wide range in mass

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in any essential particular. Thus the first two differences in the first division of Table III would be +0.″002 and −0.″002 for seventeen and thirty-seven stars, respectively. The differences in the third division would not be changed by the inclusion of the six stars also occurring in the second list.

will be included, but since there is a practicable lower limit to a, only the most massive among the more distant stars will appear in our lists.

Another and perhaps even more important form of selection arises from the fact that only the most luminous of the distant stars get into our lists; the others are too faint to be seen. This, too, excludes in succession the smaller masses with increasing distance, and is related to the correlation of mass with spectral type discussed in section 16.

The influence of distance on the selection of mass seems to be shown by the data now available. To test the matter, consider the sixty-nine binaries of known parallax[1] in the first list of Jackson and Furner. For these, the individual masses, μ, can be calculated by (4). Comparing these values with the mean mass for stars of the same spectrum, as interpolated from Table IV, we obtain a series of residuals $\Delta \log \mu = \log \mu - \log \underline{\mu}$. When grouped according to parallax, these lead to the mean systematic deviations shown in Table V. A progression with increasing parallax, in the direction anticipated, is evident, the negative sign of the differences associated with the more distant stars indicating that the corresponding

TABLE V

DEPENDENCE OF MASS ON PARALLAX

π_s	Mean Sp.	No. Stars	$\overline{\Delta \log \mu}$	Curve
0″018	F2	10	−0.08	−0.11
.026	F6	10	−0.12	−0.08
.032	G1	10	−0.08	−0.07
.046	F7	10	+0.03	−0.03
.061	G3	10	−0.04	0.00
.102	G1	10	+0.04	+0.04
0.270	G8	9	+0.12	+0.12

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[1] The thirty-six stars of known parallax in the second list are not included because of the systematic difference in the parallaxes of the two lists. In the preparation of Table II, where only a mean result for all the stars was finally used, the systematic difference can be corrected with sufficient precision. Here, however, we are concerned with individual stars, or at most with small groups, and it is not certain that the mean correction is applicable. Then, too, the fact that the second list of Jackson and Furner is based on a statistical formula, renders these data unsuitable for the present discussion.

mean masses in Table IV are too large. The character of the progression is best seen from the smoothed values of the differences given in the last column of Table V. Considered as corrections, these differences are such as would reduce the masses to values representing the selection for $\pi = 0''.06$. Even at this moderate distance the smallest masses will still be excluded; hence some further correction, a constant, is required. This cannot be accurately specified; but in general it may be said that, if we regard the selection for the two nearest groups in Table V as approximately representative of the visual binaries as a whole, the data of this table would indicate that the mean masses of the binaries actually observed at distances corresponding to $\pi = 0''.02$ are roughly 50 per cent larger than those of a representative collection.

This result at once suggests that the variation of log $\underline{\mu}$ with spectral type found above has been influenced by selection, for in general the mean parallax of the earlier-type stars, which are all highly luminous, is less than that of the late-type dwarfs. Forming the mean parallax for each of the spectral groups in the first half of Table II, we find in fact a progression in distance as shown in Table VI. If the data of Table V may be accepted, the values of log $\underline{\mu}$ in Table IV must be corrected as indicated in the last column of Table VI, in order to obtain the true variation of log $\underline{\mu}$ with spectral type. To make the results fully representative, the additional constant correction referred to above must also be applied. This seems to be of the order of -0.08.

TABLE VI

CORRECTIONS TO LOG $\underline{\mu}$

(To Eliminate Selection in Mass)

Sp.	π_s	No. Stars	Correction to log $\underline{\mu}$
A7	$(0''.133)$	3	-0.03
F2	0.051	14	-0.02
F5	0.038	13	-0.05
G0	0.055	18	-0.01
G4	0.069	10	0.00
K2	0.114	7	$+0.06$
K8	0.229	4	$+0.11$

Unfortunately, however, the results in Table V are not as reliable as could be wished. The individual values of the masses are affected by large uncertainties, and the number of stars in the separate groups is small. Further, the residuals $\log \mu - \log \underline{\mu}$ depend upon the ordinates of the curve for ΔM shown in Figure 1. Any error in the curve, in the nature of a tilt with respect to the axis, must therefore have introduced a progression in the residuals which only partially disappears when they are grouped according to parallax, because such a grouping, in part at least, is also a grouping with respect to spectral type. This uncertainty is probably not very serious, for Table V shows that the relative effect on the corrections to $\log \mu$, for the two extremes of distance, is the differential error in the ordinates of the ΔM curve for F2 and G8. Since these ordinates are not widely separated, the influence of incorrect slope in the curve cannot be great.

6. ADOPTED MASSES

On the basis of the evidence in Table VI, the corrections given in the sixth column of Table IV were finally adopted. These are the smoothed values of the results in Table VI, extrapolated to B0 and slightly modified in the case of type K because of the small number of these stars. The small constant correction referred to above is ignored, because of its uncertainty and relative unimportance. It merges with the uncertainty involved in the assumptions of comparability which must be made in extending the results for the masses of the binaries to the stars at large.

The differential correction, affecting the relative values of the masses as it does, is of more interest. The present determination is to be considered as only a first approximation; the results are at least in the right direction, and it is believed that they do not represent an over-correction.

We have, finally, as adopted values of the geometrical mean masses of the visual binaries, the data in the seventh and eighth columns of Table IV. These results refer mainly to dwarf stars and correspond to the mean absolute magnitudes given in the second column of the table.

Owing to the small number of late-type giants in the lists of visual binaries, it is not possible to use the statistical method for a determination of the masses of these stars. Some indication of their values is afforded, however, by the application of equation (4) to such of these objects as occur in the spectroscopic list. For 14 G and K stars, with a mean absolute magnitude of 0.0,

$$\log \underline{\mu} = 0.47 \pm 0.16$$

whence $\underline{\mu} = 3.0$ with an uncertainty of about 40 per cent.

The use of the geometrical mean makes it difficult to compare the masses thus found with other results,[1] for which the arithmetical mean has usually been employed. For a Gaussian distribution of the logarithms of the mass, which seems to be that indicated by the available data, the conversion formula would be[2]

$$\log \bar{\mu} = \log \underline{\mu} + 2.53\, r^2$$

where $\bar{\mu}$ is the arithmetical mean, and r the probable dispersion, i.e., the median value of the differences $\log \mu - \log \underline{\mu}$, arranged in order without regard to sign. The value of r is difficult to determine. For the dwarf stars it is certainly small, probably not larger than 0.15 (see page 185), whence, approximately, $\bar{\mu} = 1.14\, \underline{\mu}$.

Applied to the present data, this relation would lead to arithmetical mean masses a little larger than those previously found, but close comparisons are not possible. The change in mass with spectral type is of the general character of that resulting from other investigations and is certainly well established, although the variation given here is not as large as that derived by Ludendorff[3] from a discussion of spectroscopic binaries.

Strictly speaking, the present results refer only to the particular group and class of stars upon which the conclusions are based.

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[1] Russell, *loc. cit.; Publications of the American Astronomical Association*, **3**, 327; *Popular Astronomy*, **25**, 666, 1917. Aitken, *The Binary Stars*, pp. 206 ff., 1918. Van Maanen, *Publications of the Astronomical Society of the Pacific*, **31**, 231, 1919. Ludendorff, *Astronomische Nachrichten*, **189**, 145, 1911; **211**, 105, 1920.

[2] Adams, Strömberg, and Joy, *Mt. Wilson Contr.*, No. 210, p. 17; *Astrophysical Journal*, **54**, 25, 1921. Although derived for use in another connection the formula is generally applicable.

[3] *Loc. cit.*

Their applicability to any other class of stars presupposes some assumption. Thus in passing from the combined masses of the visual binaries to those of single stars, we can make use of the average mass-ratio of the two components and associate the resulting average values for the central stars with that for the non-binaries of the same spectra; but it is by no means certain that the stars so associated are strictly comparable. For the further discussion we make this assumption. Presumably the uncertainty is not large, for we thus arrive at a mean mass for the solar-type stars which is the same as that of the sun itself. To give this conclusion force it must be added that the average deviation of the mass of a dwarf star from the mean mass for the type in question is small (see next section).

There is some evidence that the mass-ratio varies with the type.[1] Adopting, however, the mean value 0.75 for the ratio of secondary to principal component, we find for the stars at large the geometrical mean masses ($\mathcal{M}$) in the last two columns of Table IV.

7. DISPERSION IN MASS

The residuals in log μ for the sixty-nine stars discussed above throw some light on the dispersion in mass. Freeing the individual residuals from the influence of selection in mass, we find the frequencies shown in Table VII. In spite of the small numbers of

TABLE VII

DISTRIBUTION OF Δ LOG μ

INTERVAL OF Δ LOG μ	NO. RESIDUALS		TOTAL	
	+	−	Obsd.	Theor.
0.00–0.09	8	4	12	17
0.10–0.19	10	8	18	15
0.20–0.29	6	7	13	12
0.30–0.39	5	7	12	10
0.40–0.49	5	5	10	6
0.50–0.59	0	1	1	4
0.60–0.69	1	1	2	3
0.70–0.79	0	0	0	1
0.80–0.89	1	0	1	0
Sums	36	33	69	68

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[1] Aitken, *The Binary Stars*, pp. 206, 216, 1918.

stars, it is seen from the second and third columns that the frequency curve is approximately symmetrical. Further, the last two columns show that, within the uncertainty of the data, the values of log μ follow a Gaussian distribution.

The probable dispersion (dwarfs Fo to M) is

$$r(\Delta \log \mu) = \pm 0.22. \tag{10}$$

This, however, is not a true measure of the dispersion in log μ, for the result has been derived with the aid of equation (4) and includes, therefore, the influence of errors in M and M_c. Those in M_c arise from the factor a^3/P^2 and are relatively small. Bearing on the accuracy of M are the data in *Contribution* No. 199, p. 15, from which for the spectral types and luminosities involved, $r(M) = \pm 0.36$ mag. By (4), the contribution to the calculated dispersion in log μ arising from errors in M is therefore ± 0.22, which is the same as the calculated value of $r(\Delta \log \mu)$ itself. Since the dispersion in the masses presumably is not zero, the adopted value of $r(M)$ is probably too large. This unfortunately makes it impossible to gain any definite information as to the dispersion, and we can only indicate, as in the second column of Table VIII, the probable values corresponding to various assumed values of $r(M)$. The last column of the table gives the limits within which it is an even chance that the mass of any visual binary will be found. Thus, if we assume that the true probable error of the spectroscopic absolute magnitude is ± 0.25, the masses of one-half of the stars should be between 0.7 and 1.5 times the mean mass.

TABLE VIII

PROBABLE DISPERSION IN MASS

$r(M)$	Probable Dispersion in Log μ	Limits for Prob. = ½
±0.35	±0.07	$0.85\mu - 1.18\mu$
0.30	0.13	0.74 – 1.35
0.25	0.16	0.69 – 1.45
0.20	0.18	0.66 – 1.51
±0.15	±0.20	0.63 – 1.58

Apparently the range in the masses of the dwarf stars from Fo on must be small. Further evidence is given on page 214. These results are in general agreement with the earlier findings of Russell and Eddington.[1]

8. EQUIPARTITION OF ENERGY

A possible analogy between the dynamical behavior of the stars and that of the molecules of a gas has frequently been mentioned.[2] Applicability of the kinetic theory would imply an equipartition of the energy of translation such that

$$\mu_1\overline{v_1^2}=\mu_2\overline{v_2^2}= \ . \ . \ . \ =\mu_n\overline{v_n^2} \tag{11}$$

where the individual values in $\overline{v_n^2}$ include all the velocities associated with a definite class of masses μ_n. In a gas this final state is brought about very quickly; in a collection of stars the period of relaxation, if the term has any meaning at all, obviously must be enormously long.

As a matter of fact, the theory for a collection of stars is much simpler than that of a gas, because the occurrence of collisions and close encounters is so rare that they may be neglected.[3] The effective agency for the transfer of energy is therefore only that arising from the attraction of the great mass of stars as a whole upon the individual members of the group. But this simplification of the theory and the consequent restriction of the mechanism for the redistribution of energy so enormously lengthens the interval required for the attainment of statistical equilibrium that Eddington[1] rejects altogether the analogy with a gas and proceeds on the principle that the motions of the stars do not mutually interfere.

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[1] *Monthly Notices*, 77, 30, 1916; 604, 1917. Russell, in this connection, makes the important suggestion that the real dispersion in mass may be largely concealed in a discussion based upon spectroscopic data. This would be the case if the spectral characteristics used for the determination of a star's absolute magnitude are functions of its surface brightness (temperature) and density alone. For Russell's discussion of this interesting point see the following note on page 238.

[2] See for example Poincaré, *Hypothèses Cosmogoniques*, p. 257, 1911; Halm, *Monthly Notices*, 71, 634, 1911; Eddington, *Stellar Movements*, pp. 159, 247, 1914; Jeans, *Problems of Cosmogony and Stellar Dynamics*, pp. 224 ff., 1919.

[3] Jeans, *op. cit.*, p. 229.

As far as the present state of the universe is concerned, Jeans[2] concurs in this opinion, but points out that the conditions of an earlier stage in its history may have been entirely different. Following his conjecture that the system has evolved from a rotating nebula of dimensions much less than those of its present configuration, he shows that the earlier stages of development would be favorable to a rapid redistribution of energy, and that the system may have proceeded far on the way toward equipartition before its gradual expansion reduced the rate of transfer to its present negligible amount. The fact that readjustments are now inappreciable does not therefore demonstrate the absence of equipartition, which must be put to an observational test.

Although a certain type of systematic motion is not incompatible with a steady state, the phenomenon of star-streaming shows that equipartition has not been completely attained. Within the individual streams, however, there may be an approach to this state, just as in the case of two intermingling streams of gas. Further, if the stream motion is small as compared with the peculiar velocities of the stars, it will have little influence on the statistical averages in (11). For a decade or more some correlation of mass with velocity of the type hereby implied has been suspected, for it has been known that in general the most massive stars are the most luminous, and the most luminous stars have on the average the smallest velocities.[3] Now, however, we are in possession of data which admit of a more definite test.

For the B and A stars, whose extreme range in luminosity is small, we have the determinations of mean radial velocity by

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[1] Eddington, *loc. cit.*, p. 254. See also *Astronomische Nachrichten, Jubiläumsnummer*, p. 9, 1921.

[2] *Op. cit.*, p. 237.

[3] Campbell, *Stellar Motions*, p. 207, 1913. Kapteyn, *Mt. Wilson Contr.*, No. 45; *Astrophysical Journal*, **31**, 258, 1910. Russell, *Observatory*, **37**, 174, 1914. Kapteyn and Adams, *Mt. Wilson Comm.*, No. 1; *Proc. Nat. Acad. of Science*, **1**, 14, 1915. Adams and Strömberg, *Mt. Wilson Contr.*, No. 131; *Astrophysical Journal*, **45**, 293, 1917.

Campbell.[1] For the remaining types, within which the differences in luminosity are much greater, we have the recent Mount Wilson results,[2] giving both radial and space velocities for each of these types as a function of absolute magnitude. Since the values of the masses in the last column of Table IV are probably representative of the stars at large, they may be combined with the mean velocities of the different types, in conformity with (11), to determine the degree to which the condition of equipartition is satisfied.

The values of the velocities have not been corrected for stream motion and thus subject the hypothesis of equipartition to a severe test. The Mount Wilson results for radial and space velocities include the K-term. For the late-type dwarfs, this is small, and negligible for the present discussion. Campbell, on the other hand, has freed his results from this term, which for the B stars is two-thirds of the mean radial velocity itself. If one prefers to regard the K-term as a real velocity, and not merely a systematic correction, its contribution to the energy can be found as indicated below.

It has been shown that the distribution of the space velocities for types F to M is not in accordance with Maxwell's law,[3] but agrees well with the assumption that the logarithms of the velocities follow a Gaussian distribution. Nevertheless, Campbell's theorem to the effect that the arithmetical mean space velocity equals twice the mean radial velocity, holds for the giants with close approximation.[4] We assume that a similar state of affairs exists for the B and A stars, and thus obtain for these types the arithmetical means of their space velocities. The square of the arithmetical mean space velocity could have been used directly for a test of equipartition; but to evaluate the influence of the K-term, considered as a velocity, it is simpler to use the mean-square space velocity.

[1] *Lick Observatory Bulletins*, **6**, 101, 1911; **7**, 19, 1912.

[2] Adams, Strömberg, and Joy, *Mt. Wilson Contr.*, No. 210; *Astrophysical Journal*, **54**, 9, 1921. These results are vital for the remainder of the discussion. Without them it could not have been undertaken.

[3] Kapteyn and Adams, *Mt. Wilson Comm.*, No. 1; *Proc. Nat. Acad. of Sciences*, **1**, 14, 1915. Adams, Strömberg, and Joy, *op. cit.*, p. 15.

[4] *Ibid.*, p. 13.

For a Gaussian distribution of the logarithms of the space velocities, the relation of the mean square velocity to the arithmetical and geometrical means is expressed by[1]

$$\left.\begin{aligned}\log \overline{v^2} &= 2\log \bar{v} + \frac{1}{2h^2\,\text{Mod.}} = 2\log \bar{v} + 0.148\\ \log \overline{v^2} &= 2\log \underline{v} + \frac{1}{h^2\,\text{Mod.}} = 2\log \underline{v} + 0.296\end{aligned}\right\} \qquad (12)$$

where *h* is the modulus of the distribution function. It is assumed that the value of *h* for types F to M given in *Contribution* No. 210 applies also to the B and A stars.

The mean radial velocities taken from Campbell's papers[2] are

B3, 6.2; B8.5, 6.7; A0, 10.3; A2, 11.4; A5, 12.6.

The corresponding mean square velocities were obtained by doubling the radial velocities, as indicated above, and substituting the results into the first of (12). For the later types the values of $\log \overline{v^2}$ were derived from the data for $\underline{v}$ given in *Contribution* No. 210. These were plotted—$\underline{v}$ against corresponding M—thus giving four curves, which refer to the approximate mean types F5, G5, K3, and Ma+. The mean absolute magnitudes of the dwarf stars of these types were then interpolated from Table IV and used as arguments to read from the curves values of $\underline{v}$ corresponding to the average dwarf of each of the four types. Graphical interpolation and an extrapolation to F0, followed by substitution into the second of (12), gave finally the required values of $\log \overline{v^2}$ for every

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[1] The second equation is found by applying the mean value theorem to the frequency function

$$F(v) = \frac{h\,\text{Mod.}}{\sqrt{\pi}}e^{-h^2(\log v - \log \underline{v})^2}$$

The first results from the combination of the second with

$$\log \bar{v} = \log \underline{v} + \frac{1}{4h^2\,\text{Mod.}}$$

which is given by Adams, Strömberg, and Joy, *op. cit.*, p. 17.

[2] *Lick Observatory Bulletins*, 6, 127, 1911; 7, 28, 1912.

half-interval of spectral type from F0 to Ma. These mean-square velocities are not affected by any smoothing from type to type, since the interpolation card was drawn *through* the points derived from the four curves giving v the function of M.

The results, including those for log $\mathcal{M}\overline{v^2}$, are in the first six columns of Table IX. With the exceptions of the B stars, the

TABLE IX

THE ENERGY CONSTANT

Sp.	M	Log $\mathcal{M}$	Log $\overline{v^2}$	Log $\mathcal{M}\overline{v^2}$	Residual	Log θ^2	Log $\mathcal{M}\theta^2$	Residual
B3	−0.6	0.95	2.34	(3.29)	(+0.28)	1.74	(2.69)	(+0.07)
B8.5	+0.4	0.81	2.40	(3.21)	(+ .36)	1.62	(2.43)	(+ .33)
A0	0.7	0.78	2.78	3.56	+ .01	2.00	2.78	− .02
A2	1.0	0.70	2.87	3.57	.00	2.10	2.80	− .04
A5	1.5	0.60	2.95	3.55	+ .02	2.19	2.79	− .03
F0	2.4	0.40	3.11	3.51	+ .06	2.36	2.76	.00
F5	3.3	0.19	3.36	3.55	+ .02	2.56	2.75	+ .01
G0	4.4	9.99	3.62	3.61	− .04	2.76	2.75	+ .01
G5	5.2	9.88	3.78	3.66	− .09	2.84	2.72	+ .04
K0	5.9	9.83	3.80	3.63	− .06	2.84	2.67	+ .09
K5	7.1	9.79	3.74	3.53	+ .04	2.92	2.71	+ .05
Ma	9.8	9.77	3.78	3.55	+0.02	3.06	2.83	−0.07
Means, average deviations				3.57	±0.036	......	2.76	±0.036

values of the energy show close accordance. For a range of 10 to 1 in the masses the average deviation in $\mathcal{M}\overline{v^2}$ is 9 per cent, and, by a moderate amount of smoothing, is easily reduced to 3 per cent. The large divergence for the B stars seems to be real. It cannot be supposed that the adopted values for their masses are in error by 50 per cent, nor does it seem permissible to assume that the velocities could by any chance be increased by the 2 or 3 km necessary to bring the energy up to the mean for the other types. It is more likely that the velocities are already too large, because of undetected spectroscopic binaries among the stars combined in the means.

The inclusion of the K-term as a velocity removes only a small part of the divergence. Thus, for any radial direction, the resultant mean-square velocity $\overline{r^2}$ would be

$$\overline{r^2}=\overline{v^2}+K^2-\overline{2vK\cos a}=\overline{v^2}+K^2$$

where a is the inclination of the space velocities to the given line of sight. Extending this to all radial directions, we have for the entire group of B3 stars,

$$\overline{r^2}=(14.8)^2+(4.7)^2=241.1;\ \log \mathcal{M}\overline{v^2}=3.33,$$

an increase of only 0.04 in the logarithm of the energy.

The foregoing results depend upon the assumptions that the mean space velocities of the B and A stars may be found by doubling the mean radial velocities, and that the modulus h applies equally to all types. We may avoid these assumptions by restricting the test to the mobility in the line of sight. For this comparison the K-term has been included throughout. The mean radial velocities from Campbell's data thus become[1]

B3, 7.45; B8.5, 6.50; A0, 10.1; A2, 11.3; A5, 12.5 km/sec.

For the remaining types *Contribution* No. 210 was again used, the values being read from curves for radial velocity θ as a function of absolute magnitude as in the case of $\underline{v}$. The values of log $\mathcal{M}\theta^2$ (eighth column, Table IX) show about the same degree of accordance as those of log $\mathcal{M}\overline{v^2}$, the average deviations, without smoothing, again being 9 per cent. The inclusion of the K-term as a part of the radial velocity has brought the B3 stars into agreement with the mean for types A to Ma, and only the B8–B9 stars are now divergent.[2] But this improved agreement is not to be taken very seriously, and perhaps the most satisfactory point of view is that since the B stars are peculiar in many particulars, the abnormality shown by Table IX should not be looked upon as surprising,

[1] *Ibid.*, 6, 108. The means for the B stars are those found by correcting Campbell's values V_1 by 4.7 (B0–B5) and 4.1 (B8–B9). For the A stars the mean of V_2, *Lick Observatory Bulletin*, 7, 20, has been used without correction.

[2] Freundlich, *Physikalische Zeitschrift*, 20, 561, 1919, has attempted to show that the K-term may be identified with the displacement of spectral lines toward the red required by the theory of relativity. This displacement may be involved, but results given later in this paper indicate that other factors must also play a part. See footnote p. 212.

especially when it is remembered that they form a local aggregation[1] and that their masses are among the largest known.

There are suggestions of systematic divergencies in both sets of residuals, but by forming the means of the two series of values of the energy constant, these disappear, and the average deviation is reduced to 5 per cent.

Were the condition of equipartition exactly satisfied, we should have

$$\overline{v^2}=3\,\overline{\theta^2}\,.$$

For a Gaussian distribution of the radial velocities

$$\overline{\theta^2}=\frac{\pi}{2}\,\bar{\theta}^2$$

where $\bar{\theta}^2$ is the square of the arithmetical mean radial velocity used in Table IX. For unit mass, $\log \overline{v^2}=3.57$; and since $\log \pi/2=0.20$, $\log \overline{\theta^2}=2.96$. The mean-square velocities therefore appear to have a ratio of 4 to 1 instead of 3 to 1. It is known, however, that the radial velocities do not follow a Gaussian distribution closely.

Approaching the matter from another direction, we have for a random distribution in direction, irrespective of the form of the frequency function of the magnitudes of the velocities,

$$\bar{v}=2\,\bar{\theta}.$$

Substituting into the first of equations (12), we find the condition

$$\log \overline{v^2}=\log \theta^2+0.75$$

where θ^2 denotes the square of the arithmetical mean. The difference in the two members of this equation is 0.06 in the logarithm, or 11 per cent, a much better agreement than before. The remaining difference is doubtless to be attributed to the fact that the velocity distribution is not random; stream motion produces a preferential drift parallel to the galactic plane.

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[1] This result follows directly from an analysis of star counts. Seares, *Publications of the Astronomical Society of the Pacific*, **30**, 114, 115, 1918. Shapley in recent papers has accumulated much additional evidence: *Mt. Wilson Contr.*, Nos. 157, 161; *Astrophysical Journal*, **49**, 311, 1918, and **50**, 107, 1919; *Mt. Wilson Communications*, Nos. 54, 64; *Proceedings of the National Academy of Sciences*, **4**, 224, 1918; **5**, 434, 1919.

In spite of the known departures from Maxwell's law, there is, with the exception of the B stars, a surprisingly close approach to a state of equipartition. That the deviations are no larger is remarkable,[1] and suggests that for the stars considered (dwarfs) there is little difference between the different spectral types in the average contribution of stream motion to the total energy of translation. Since the unit of mass—that of the sun—is 1.9×10^{33} grams, and since the velocities are expressed in kilometers per second, the mean kinetic energy of translation is of the order of 3.5×10^{46} ergs.

9. DERIVATION OF MASS FROM THE ENERGY-CONSTANT

The constancy of the energy is apparently such as to justify an attempt to evaluate the masses of stars having luminosities and velocities different from those appearing in Table IX. Thus by reading from the curves referred to above the values of $\underline{v}$ and θ corresponding to different values of M and combining them with the energy-constant, more or less hypothetical values of the corresponding masses can be obtained.

This procedure assumes that equipartition holds for the stars of high luminosity as well as the dwarfs, for which alone a direct test has been possible. The behavior of the B stars suggests caution in an extension of the principle to other giants; but since the translatory energy of the B's is of the same order as that found for the dwarfs, it is perhaps not too much to expect that the use of the energy constant will lead to masses for later-type giants which will also be of the right order of magnitude.

The results obtained in this manner are given in Table X. The values of log $\mathcal{M}$ are the means of the two values found with the aid of $\underline{v}$ and θ from the two energy constants, log $\mathcal{M}\overline{v^2}$ and log $\mathcal{M}\theta^2$. To indicate the agreement between the masses derived with the aid of the two velocities, the differences log $\mathcal{M}_v -$log $\mathcal{M}_\theta$ are also given in the table. In general these are reasonably small, although for the group of Ma stars the extreme values correspond to a two-to-one ratio in the masses. The data of Table X express, with more detail and precision, the correlation between mass and luminosity found by van Maanen.[1]

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[1] In this connection see Jeans, *Problems of Cosmogony*, p. 238, 1919.

TABLE X

MASS AND LUMINOSITY

M	F5		G5		K3		Ma+	
	log $\mathcal{M}$	Diff.	log $\mathcal{M}$	Diff.	log $\mathcal{M}$	Diff.	log $\mathcal{M}$	Diff.
−3	......	....	0.74	−0.08	......	....	0.98	−0.36
−2	0.78	−0.02	.62	+ .02	......	....	.64	− .02
−1	.74	− .02	.51	+ .08	0.72	−0.10	.41	+ .16
0	.68	− .02	.38	+ .10	.45	+ .08	.23	+ .24
+1	.62	+ .02	.28	+ .14	.28	+ .14	0.08	+ .30
2	.51	.00	.17	+ .11	.17	+ .12	......	....
3	.30	− .02	0.06	+ .06	.07	+ .08	......	....
4	0.01	−0.04	9.96	− .02	0.01	+ .04	......	....
5	9.77	....	9.88	− .11	9.94	− .02	......	....
6	......	....	9.81	−0.21	9.89	− .04	......	....
7	......	....	......	....	9.84	− .06	......	....
8	......	....	......	....	9.81	−0.08	......	....
10	......	....	......	....	......	....	9.68	+0.14

The general relation of these results to spectral type and absolute magnitude is best seen by means of the diagram in Figure 2. This was prepared by plotting the data in Table X and reading from the resulting curves the values of M coresponding to integral values of the mass $\mathcal{M}$. These values of M were then plotted in the diagram of Figure 2, along with similar points on the dwarf branch, whose co-ordinates were derived from the second and last columns of Table IV. Through these points were drawn the full-line curves of the figure, which thus represent lines of equal mass.

The curves are highly irregular, the maximum near G0 followed by the minimum at K0–K5 being particularly noteworthy. Much of this disappears, however, when the ordinary visual absolute magnitude is replaced by the bolometric magnitude, which measures the total radiation and is therefore the quantity of physical significance. The reductions to bolometric magnitude (see following section) of interest here are

G0, −0.09; G5, −0.30; K0, −0.73; K5, −1.30; Ma, −1.60 mag.

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[1] *Publications of the Astronomical Society of the Pacific*, **31**, 231, 1919.

The application of these corrections would practically remove the minima from the curves for all but the largest masses, and these

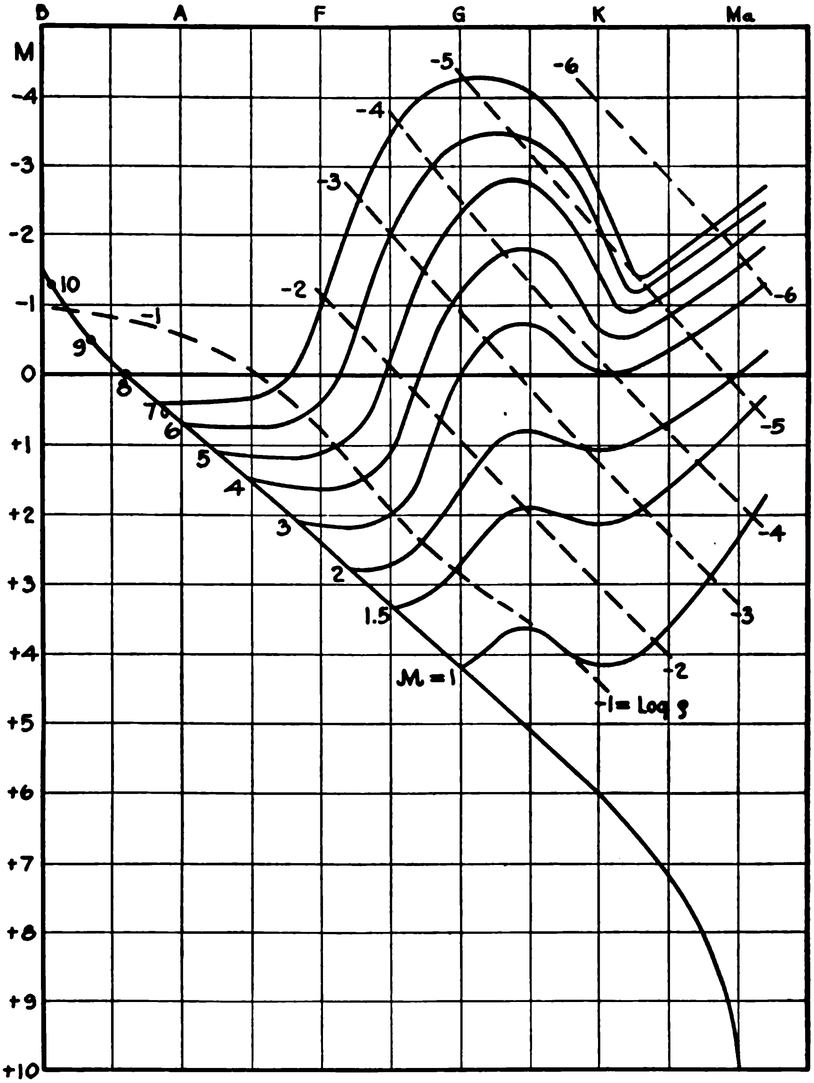

FIG. 2.—Distribution of mass (full-line) and mean density (broken curves) derived from the principle of equipartition. The full-line curve running downward to the right is the line of the maximum frequency of the dwarfs.

are relatively uncertain because they depend more or less upon extrapolations of the data in Table X.

To explain the derivation of the lines of equal density, which are also shown in Figure 2, it is necessary to consider the question of surface brightness.

10. SURFACE BRIGHTNESS

To derive values of surface brightness we may start from Stefan's law

$$E=\sigma T^4 \tag{13}$$

where E represents the total energy radiated per unit area of the surface of a star whose effective temperature is T. Let J be the integrated visual intensity, referred to the same unit of surface, and write $J=aE$, whence

$$2.5 \log E=2.5 \log J-2.5 \log a$$

and, by substitution into (13),

$$2.5 \log J-2.5 \log a=2.5 \log \sigma+10 \log T.$$

Further, let M_B and j represent the total energy and surface brightness, respectively, expressed in magnitudes. The former is the bolometric absolute magnitude, while $M=-2.5 \log (J\times \text{Area})$ is the ordinary visual absolute magnitude. Since a is also the ratio of the visual energy to total energy received from the entire surface of the star,

$$-2.5 \log a=M-M_B$$

and

$$j=M-M_B-10 \log T+\text{const.} \tag{14}$$

where the constant involves $\log \sigma$ and an additional constant depending on the zero point chosen for M_B. Eddington[1] has calculated values of the difference between the visual and bolometric absolute magnitudes $M-M_B$ for a range of temperatures including the normal classes of spectra. Through these values, which are based upon Nutting's measures of visual sensibility for the normal eye, j, by (14), becomes a simple function of T.

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[1] *Monthly Notices*, 77, 605, 1917. Nutting's measures of visual sensibility, which are the basis of Eddington's calculation, are practically identical with the A.I.E. visibility-curve.

Results found by this equation may be checked by an equivalent formula derived by Hertzsprung in 1906. Starting from Planck's law and the visibility measurements of Langley, Abney, and others, he found[1]

$$j_H = +2.3\left(\frac{c}{T}\right)^{0.93} + \text{const.} \qquad (15)$$

in which c is the second constant in Planck's formula, for which the value adopted is 14300. Both (14) and (15) involve quadratures, and, starting as they do from entirely different observational data, independently discussed, provide an excellent numerical control. How close the agreement is may be seen from Table XI.

TABLE XI

SURFACE BRIGHTNESS AND TEMPERATURE

T	$M-M_B$	j	j_H	$j-j_H$
2540°	+2.59	+6.32	+6.32	0.00
3000	+1.71	+4.72	+4.68	+ 4
3600	+0.95	+3.17	+3.16	+ 1
4500	+0.35	+1.60	+1.58	+ 2
6000	0.00	0.00	0.00	0
7500	+0.02	−0.95	−0.93	− 2
9000	+0.12	−1.64	−1.62	− 2
10500	+0.31	−2.12	−2.09	− 3
12000	+0.53	−2.48	−2.45	−0.03

The values of $M-M_B$ have been taken directly from Eddington's table. The maximum difference between the results from the two formulae is only 0.04 mag., which is ample.

To connect these results with spectral type, Wilsing's spectral-photometric measures of the reciprocal temperature, c/T, of 199 stars[2] were plotted against their respective types. The ordinates (c/T) of the curve thus obtained are in the second column of Table XII. The corresponding effective temperature, T, and the surface brightness, j, calculated by the foregoing formulae and expressed in magnitudes ($\odot = 0.00$), are in the third and fourth columns, respectively.

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[1] *Zeitschrift für wissenschaftliche Photographie*, 4, 43, 1906.

[2] *Potsdam Publikationen*, No. 74, 1919.

With few exceptions, the stars in Wilsing's list are all giants whose mean absolute magnitudes, with sufficient approximation, may be assumed to be zero. To extend the results to other luminosities, use was made of the Mount Wilson color indices,[1] whose values are in the fifth and last columns of Table XII, those for the dwarfs corresponding to the mean absolute magnitudes in the column immediately preceding. The color indices were reduced to Schwarzschild's absolute system,[2] whose relation to effective temperature is known. This gave the relation between the Mount Wilson values and the temperature for both giants and dwarfs. Owing to various necessary assumptions the temperatures thus found for the giant stars differ slightly from the adopted values given in the first half of Table XII, and this systematic difference was made the basis for reducing the calculated temperatures for the dwarfs to the adopted system of Wilsing. The results, in

TABLE XII

SPECTRUM, EFFECTIVE TEMPERATURE, AND SURFACE BRIGHTNESS

Sp.	GIANTS, $M = 0$				DWARFS				
	c/T	T	j	MW CI	c/T	T	j	M	MW CI
Bo	1.36	10500°	−2.28	−0.32	......	......	......	......	......
B5	1.43	10000	2.14	−0.17	......	......	......	......	......
Ao	1.55	9230	1.89	0.00	......	......	......	......	......
A5	1.76	8110	1.45	+0.19	......	......	......	......	......
Fo	2.04	7000	0.88	0.38	......	......	......	......	......
F5	2.35	6080	−0.26	0.62	2.35	6080	−0.26	+3.3	+0.62
Go	2.70	5300	+0.44	0.86	2.48	5770	0.00	4.4	0.72
G5	3.10	4610	1.24	1.15	2.60	5500	+0.25	5.2	0.83
Ko	3.70	3860	2.41	1.48	2.93	4880	0.90	5.9	0.99
K5	4.37	3270	3.71	1.84	3.47	4120	1.96	7.1	1.26
Ma	4.65	3080	4.25	1.88	4.30	3330	+3.58	+9.8	+1.76
Mb	4.82	2960	4.58	+1.88	......	......	......	......	......
Mc	4.94	2890	+4.81	......	......	......	......	......	......

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[1] These results, which have not hitherto been published, are still to be regarded as provisional. They take account of the important difference in color between giants and dwarfs, which cannot be disregarded in any discussion of the surface brightness of late-type stars. See *Mt. Wilson Comm.*, No. 59; *Proc. Nat. Acad. of Sciences*, 5, 232, 1919.

[2] *Göttingen Aktinometrie*, B, 29, 1912.

the sixth column of Table XII, were then used to compute the corresponding values of j as before.[1]

The difference in surface brightness between giants and dwarfs is large, and cannot be neglected. Unfortunately, the change in j with absolute magnitude is not well determined, for this must be inferred from the behavior of the color indices, and the data available at present do not extend to stars of the highest luminosity. This means that extrapolation will be necessary, and, in view of the consequent uncertainty, a simple linear relation of surface brightness to absolute magnitude has been used in the calculation of j for all values of M not appearing directly in Table XII.

The reliability of the foregoing results depends upon the possibility of representing the energy distribution in the spectra of stars of different types by means of the black-body radiation formula. It should be noted that it is not at all a question of agreement of the effective temperature, calculated from the radiation formula, with the true mean temperature of the radiating layers in the star's atmosphere. In general these will not agree. The stars are not perfect radiators; further, general and selective absorption deforms the energy-curve to such an extent that effective temperature cannot be an accurate measure of true temperature; but in the visible region the resultant curve seems to agree[2] approximately with a black-body curve, although not with the curve corresponding to the true mean temperature. That being the case, the surface brightness is immediately calculable by the foregoing formulae, provided the effective temperature corresponding to the deformed curve is employed for the computation; and it is of course just this temperature which is given by Wilsing's observations.

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[1] The values of j in Table XII may be compared with Öpik, *Astrophysical Journal*, **44**, 296, 1916; Russell, *Publications of the Astronomical Society of the Pacific*, **32**, 307, 1920. Related questions are discussed by Wilsing, *Potsdam Publikationen*, No. 76, 1920; *Astronomische Nachrichten*, **214**, 185, 1921.

[2] See, for example, Wilsing, *Potsdam Publikationen*, No. 74, pp. 17, 18. The mean systematic deviations of the observed values of the energy at different points in the visual spectrum from Planck's formula are small. Further, Milne, *Monthly Notices*, **81**, 378, 1921, finds that the theoretical spectrum for non-selective absorption is practically identical with that corresponding to black-body radiation, the principal difference being a bodily displacement of the energy-curve, corresponding to a reduction of 3 per cent in the constant of Wien's displacement law.

The possibility of systematic error in the values of the temperature is more serious. Those for the B stars given in Table XII are almost certainly too small. The early-type stars in Wilsing's list are not numerous, and the dispersion in the values of c/T is large. In fact, the data would be almost equally well represented by assigning a temperature of 12,000° to B0, which would change j from -2.28 to -2.67. The value for B5 would also be slightly changed, but that for A0 can scarcely be modified without violence to the data.[1]

It will be noted that T for a G0 giant is nearly 500° lower than that for a dwarf of the same spectrum. Some question as to the reliability of Wilsing's measures has been felt because of the low value found for G0 stars as compared with the probable temperature of the sun. But the objects observed by him are giants; with proper allowance for the sun's absolute magnitude (4.85), the corresponding temperature from Wilsing's data would be about 5820°, which is not far from the values usually assigned, and indeed, agrees well with that derived by Abbot[2] from the solar constant (5860°).

A partial control of the values of j in Table XII is afforded by Pease's recent measures of the angular diameters of stars with the interferometer. In this connection, and for later use, it is convenient to insert at this point the following formulae, all easily derived, expressing relations between the various physical constants of a star: angular diameter (d), linear diameter (D), mass ($\mathcal{M}$), density (ρ), apparent (m) and absolute (M) magnitude, and surface brightness (j).

$$\log d = 0.2\,(j-m) - 2.061 \tag{16}$$

$$\log D = 0.2\,(j-m) - \log \pi - 0.030 \tag{17}$$

$$\log D = 0.2\,(j-M) + 0.970 \tag{18}$$

$$D = 107.4\,d/\pi \tag{19}$$

$$j = 5\log d + m + 10.30 \tag{20}$$

$$\log \rho = \log \mathcal{M} - 3\log R + 0.14 \tag{21}$$

$$\log \rho = \log \mathcal{M} + 0.6\,(M-j) - 2.77 \tag{22}$$

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[1] See also Rosenberg, *Abhandlungen der Kaiserl. Leop.-Karol. deutschen Akademie der Naturforscher*, 150, No. 2, 1914.

[2] *Op. cit.*, p. 114.

These formulae give d in seconds of arc, D and $\mathcal{M}$ in terms of the solar diameter and mass, and j in magnitudes, the value for the sun being 0; the unit for ρ is the density of water. The numerical constants depend upon the following data for the sun: $D=865{,}000$ miles, $d=1920''$, distance from earth 92,930,000 miles, $m=-26.72$, $M=+4.85$. Equations (16) and (17) are the equivalent of formulae already given by Russell.[1]

Proceeding now to a comparison with the results of Pease's measures, we have the data in Table XIII. The "observed" values of j derived from the angular diameter by (20) are in the sixth line of the table. Immediately below are the calculated results interpolated from Table XII. It seems unlikely that the mean systematic difference for O−C will exceed one- or two-tenths of a magnitude. The effect of darkening at the limb has not been taken into account, either in the measures of d or in the calculation of j. Nor has the question of a possible differential effect from this source upon the observed and calculated values of j been considered, for it is evident that this cannot be large.

TABLE XIII

COMPARISON WITH MEASURED ANGULAR DIAMETERS*

	α Orionis	α Boötis	α Scorpii
Sp	Ma	Ko	Map
m	+0.9	+0.2	+1.2
π	0.″020	0.″095	0.″0085
M	−2.6	+0.1	−4.2
d (Pease)	0.″047	0.″022	0.″040
Obs. j, equation (20)	+4.6	+2.2	+4.5
Cal. j, Table XII	+4.4	+2.4	+4.5
O–C	+0.2	−0.2	0.0
Obs. D, equation (19)	252	25	506
Cal. D, equation (18)	235	27	513
Approx. $\mathcal{M}$ (Fig. 3)	8	3	15
Approx. log ρ (Fig. 3)	−6.2	−3.8	−6.6

* Michelson and Pease, *Mt. Wilson Contr.*, No. 203; *Astrophysical Journal*, **53**, 249, 1921. Pease, *Publications Astronomical Society of the Pacific*, **33**, 171, 204, 1921. The parallax of α Scorpii assumes it to be a member of the Scorpius group. The other two parallaxes are weighted mean values.

It is of interest, further, to compare the values for the linear diameters calculated from the measured angular diameter by (19) with those derived from (17) or (18) with the aid of the adopted values of j. The results are in the third and fourth lines from the bottom of table. On the whole, Pease's measures afford an excellent confirmation of the surface brightness found for late-type stars of high luminosity, and at present there seems to be no observational evidence which would justify a modification of the values adopted.[1]

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[1] *Publications of the Astronomical Society of the Pacific*, **32**, 307, 1920.

11. DISTRIBUTION OF DENSITY

The densities are now easily calculated by equation (22) with the aid of values of j from Table XII. The results for stars on the dwarf branch, including the B's, are in the fourth column of Table XIV, which also gives values for the mean linear diameters D found by (18).

TABLE XIV

LUMINOSITY, MASS, MEAN DENSITY, AND DIAMETER

(Units for $\mathcal{M}$ and D are values for sun; for ρ, that of water)

SP.	DWARF BRANCH				GIANTS, $M=0$			GIANTS, $\mathcal{M}=10$		
	M	$\mathcal{M}$	ρ	D	$\mathcal{M}$	ρ	D	M	ρ	D
B0	−1.60	10	0.04[1]	6.8	.....	....	3.2	−1.2	0.08	6
B5	−0.20	8.3	0.20	3.8	8.2	0.26	3.5	−1.8	0.03	8
A0	+0.70	6.0	0.36	2.8	7.0	0.16	3.9	−2.4	0.008	12
A5	+1.50	4.0	0.40	2.4	5.6	0.071	4.8	−3.1	0.002	20
F0	+2.40	2.5	0.40	2.0	4.3	0.025	6.2	−3.6	0.0004	32
F5	+3.35	1.5	0.39	1.8	3.2	0.0078	8.3	−4.0	0.0001	52
G0	+4.35	1.0	0.68	1.26	2.6	0.0025	11.5	−3.9	0.00002	83
G5	+5.20	0.76	1.2	0.96	2.8	0.00087	17	−3.3	0.00001	102
K0	+6.00	0.68	1.3	0.89	3.0	0.00018	28	−2.3	0.00001	107
K5	+7.20	0.62	1.4	0.83	2.6	0.000026	51	−2.2	0.000002	182
Ma	+9.80	0.59	5.4	0.54	2.0	0.0000096	66	−3.0	0.0000006	288

For the determination of the equal-density lines (dotted) shown in Figure 2, values of M were read from each mass line in the diagram for every half-interval of spectrum. The use of (22) then gave the values of log ρ corresponding to each of these points in the sequence of spectral type. Plots of these results—log ρ against spectrum—then led to a series of curves, one for each integral value of the mass (also one for $\mathcal{M}=1.5$), from which were read spectral types corresponding to integral values of log ρ. The corresponding points were marked on the diagram and connected by the dotted lines shown in the figure.

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[1] Mr. Pease has kindly placed at my disposal provisional results for three other stars. The observed and theoretical values of j and d, or D, are here also in excellent agreement, although the accidental deviations are larger than those shown in Table XIII.

Within the limits considered, and with the exception of that for $\log \rho = -1$, the curves of equal density are sensibly linear; the regularity of their spacing is remarkable. Including the line for $\log \rho = -1$ from A5 onward through the later types, the mean density corresponding to any given spectrum and absolute magnitude is represented by the formula

$$\left.\begin{array}{c}\log \rho = -1.22\,S + 0.57\,M + 1.11 \\ (S = 0 \text{ for B0, 1 for A0, etc.})\end{array}\right\} \qquad (23)$$

with a probable error of about 15 per cent, over a range in ρ extending from 0.1 to 0.000001.

The question now arises as to what means can be found for testing the validity of these results. Some control is obviously necessary, for with the exception of the masses and densities along the dwarf branch, the whole scheme of mass and density distribution depends upon the assumed applicability of the principle of equipartition. In this connection, the eclipsing variables, many of which are known to be of exceptionally low density, immediately suggest themselves as suitable for a test. The weak point, however, is that we do not know their absolute magnitudes, except in a very few cases. Any attempt to use hypothetical absolute magnitudes fails, because it involves an assumption for the mass, and this, as shown by (22), is equivalent to a direct assumption for the density.

Some general indications, however, can be obtained. With slight revisions of spectral classification, Shapley's data[1] for the densities of eclipsing variables give the following mean values:

Sp.	ρ	No. Stars
B6	0.12	15
A0+	0.20	48
F(dwarfs)	0.35	9

For these groups of B and A stars the values of $\log \rho$ are -0.9 and -0.7, respectively. From Figure 2 the corresponding values of M are seen to be -0.6 and $+0.2$. These are of the order to be expected, and are obviously possible values. The value $\rho=0.35$ for the F-type dwarfs agrees closely with that found here for F5, namely, 0.39. For the dwarfs the dispersion in absolute magnitude is so small that lack of knowledge of the luminosities gives rise to no appreciable error.

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[1] *Contributions from Princeton University Observatory*, No. 3, p. 124, 1915.

These results are all satisfactory, as far as they go, but they do not reach that part of the diagram which is involved in the greatest doubt, namely the region of very high luminosity and exceptionally low density. On the other hand, if the validity of the mass and density distribution can once be established, the densities of the eclipsing variables should give a reliable determination of their distances. Fortunately, the uncertainty can be removed to a large extent with the aid of the Cepheid variables.

12. REVISION OF MASSES AND DENSITIES BY MEANS OF CEPHEID VARIABLES

There are still difficulties connected with the pulsation hypothesis as a means of explaining Cepheid variation, although there seems little doubt that, in some form or other, it must be used to account for the changes in brightness of these stars. That being the case, the mean density must vary inversely as the square of the period. Hence[1]

$$\log \rho = -2 \log P + \text{const.} \tag{24}$$

Further, the period-luminosity relation[2]

$$M = f\,(P) \tag{25}$$

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[1] The exact form of the relation of density to period as applied to Cepheids is possibly open to some question. Emden, *Gaskugeln*, p. 451, 1907, finds for ellipsoidal oscillations of a polytropic gas sphere

$$P \propto (m\,\rho)^{-\frac{1}{2}}$$

where m is the order of the harmonic oscillation; the important vibration corresponds to $m=2$. Eddington, *Monthly Notices*, **79**, 2, 1918; 177, 1919, discusses pulsations which are symmetrical about the star's center and finds a result equivalent to (24), with the exception that the constant is replaced by a slowly varying function of the specific heats and the ratio of radiation-pressure to gravitation, the latter increasing with the mass. For a first approximation we may adopt (24).

[2] Shapley, *Mt. Wilson Contr.*, No. 151; *Astrophysical Journal*, **48**, 89, 1918.

determines the absolute magnitude. With M and the spectral type as arguments, log ρ and $\mathcal{M}$ can be interpolated from the mass-density diagram. Then,

a) The interpolated values of log ρ may be compared with those calculated by (24), the constant being adjusted to give the best representation of the densities of the entire group of stars. Since the range in ρ as determined by (24) is from about 0.1 to 0.00001, the control on the densities should be effective.

b) Having determined the constant in (24), we may write (22) in the form

$$\log \mathcal{M} = -2 \log P - 0.6\,(M-j) + \text{const.} \tag{26}$$

which expresses the mass of a Cepheid variable as a function of its period and surface brightness, for, by (25), M is a function of the period alone. The values of $\mathcal{M}$ interpolated from the diagram may be compared with those calculated by (26). This checks the distribution of mass.

c) Finally, with spectral type and log ρ calculated from (24), we may interpolate M from the diagram. The absolute magnitudes thus found may then be compared with the period-luminosity relation (25). This is not an independent test, but exhibits the results of (*a*) in a different form.

Because of the enormous range in the density, the application of these tests will mean little except in the case of stars of accurately determined spectra. This makes trouble at once, for the type of a Cepheid, in so far as determined from the hydrogen lines at least, varies with its brightness, as was first clearly shown by Pease's[1] observations of RS Boötis. Isolated determinations of spectrum based on the appearance of these lines are therefore unsuitable for the comparisons. Adams and Joy,[2] on the other hand, have shown that many spectral characteristics change little or not at all as the light of the star varies; their "estimated" spectral classes are practically independent of phase.

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[1] *Publications of the Astronomical Society of the Pacific*, 26, 256, 1914.

[2] *Mt. Wilson Comm.*, No. 53; *Proceedings of the National Academy of Sciences*, 4, 129, 1918.

The list of spectroscopic parallaxes[1] includes twenty-nine Cepheids for which estimated types are available. One of these, RY Boötis, although usually classed as a Cepheid, does not show Cepheid spectral characteristics and has an absolute magnitude at variance with that derived from the period-luminosity relation. Further, three cluster-type Cepheids, SU Aurigae (8.7), SU Draconis (9.2), and SW Draconis (10.0), and the long-period Cepheid V Vulpeculae (8.6) are so faint that the types assigned are less certain than the others. The classification of faint stars, observed with low dispersion, is likely to be influenced to some extent by the hydrogen lines. The case of V Vulpeculae, moreover, is peculiar. The light-curve is irregular; and the period-luminosity relation leads to a value of M which not only differs widely from that derived from the spectrum, but falls outside the limits of the mass-density diagram. These five stars were therefore excluded,[2] leaving a total of twenty-four.

These data are supplemented by Shapley's[3] results for twenty Cepheids, based on systematic observations of spectral characteristics that do vary with the changes in light. The median value of the type, which was used for the discussion, is about 0.3 of a spectral interval earlier than the estimated spectrum.

This difference will lead to different values of the constants in the foregoing formulae, unless the value of M and the spectral type are referred to the same phase. Thus, the estimated type agrees with the hydrogen-line type at, or very near, the time of minimum brightness (estimated *minus* hydrogen-line type at minimum $= -0.1$ spectral interval). Hence for the comparisons, the minimum, and not the median M (the period-luminosity relation gives the latter), should be associated with the estimated spectra. The simplest procedure, however, is to use the median M and subtract 0.3 of an interval from the estimated spectra before making interpolations from the diagram. Sixteen stars are common to the two lists, but owing to the different methods of classification used, the duplicate observations have been treated as separate stars.

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[1] *Mt. Wilson Contr.*, No. 199; *Astrophysical Journal*, **53**, 13, 1921.

[2] It was not noted until after the comparisons had been finished that, on the criterion of brightness, W Serpentis (9.0) should also have been excluded.

[3] *Mt. Wilson Contr.*, No. 124; *Astrophysical Journal*, **44**, 273, 1916. In general the means of the limiting values of the type as given by Shapley have been adopted. Deviations from this rule occur in a few cases where the distribution of the observations is such that the mean does not correspond to median brightness of the variable.

The results of the application of tests (*a*), (*b*), and (*c*) are summarized in Table XV, which gives the range in log ρ, log $\mathcal{M}$,

TABLE XV

REPRESENTATION OF MASSES AND DENSITIES
EQUATIONS (24) TO (26) AND FIGURE 2

TEST	RANGE	AVERAGE DIFFERENCE	
		Est. Sp.	H.L. Sp.
(*a*) log ρ	-1.1 to -4.8	± 0.15	± 0.18
(*b*) log $\mathcal{M}$	0.46 to 1.28	0.15	0.16
(*c*) M	-0.3 to -4.7	± 0.30	± 0.32

and M over which the comparisons were made, and the average differences resulting therefrom. Thus, on the basis of estimated spectra, formula (24) represents the density distribution in Figure 2 over a range in log ρ from -1.1 to -4.8, with an average difference of ± 0.15. Similarly, the values of M interpolated from the diagram in the interval -0.3 to -4.7 agree with those from the period-luminosity relation within ± 0.30 mag. We conclude at once that the masses and densities derived with the aid of the principle of equipartition are of the right order of magnitude, at least as far as type G3, which is the latest occurring among the Cepheids used.

A classification of the differences in log ρ and log $\mathcal{M}$ according to M, which is also more or less a classification according to spectrum, revealed, however, the presence of systematic divergences. These run from about -0.18 (at $M = -0.7$, A8) to $+0.18$ (at $M = -3.8$, F8) in the logarithms of both quantities. The algebraic signs mean that the range of mass in Figure 2, for a given interval in M, is too small, that the masses of the most luminous Cepheids are larger, and those of less luminous stars smaller, than the values given by the diagram. The density lines, on the other hand, are too closely spaced.

These differences were made the basis of a readjustment of the lines of mass and density. The percentage divergence in ρ and $\mathcal{M}$

is the same, as it must be by (22), but since the scale for $\mathcal{M}$ is much larger than that for ρ (in passing from one curve to the next the mass changes by one unit while ρ changes tenfold), the adjustments were made in the mass-curves, the densities being computed by (22) as before. The masses and densities along the dwarf branch, of course, remain unchanged.

The result of several trials is the diagram shown in Figure 3. A repetition of tests (*a*), (*b*), and (*c*) gave the differences shown in the last six columns of Table XVI, the first column under each of the headings log ρ, log $\mathcal{M}$, and M referring to estimated spectra, the second to those based on the hydrogen lines. The average differences have been reduced to about two-thirds their original values, while the run in the differences has disappeared. Thus for four groups of eleven stars each, the mean systematic values are:

M	−1.0	−2.1	−2.7	−3.6
Sp.	F0	F6	F7	F8
log ρ	−0.01	−0.03	+0.02	+0.02
log $\mathcal{M}$	−0.02	−0.01	−0.04	0.00

The comparison of the interpolated values of M with the period-luminosity relation is shown graphically in Figure 4.

In this connection it will be noted that values of M interpolated from the diagram with log ρ and the type as arguments are sensitive to uncertainties in the spectra. The relations are such that the average difference ±0.2 in M (Table XVI) corresponds to 0.1 of a spectral interval (see equation [23]). The normal uncertainty of classification therefore accounts for nearly all the deviation shown by individual stars in Figure 4.

The principal change in the diagram has been the depression of the mass-curves near G0 and the change in their curvature in the region of the early types. Since the data for the Cepheids do not extend beyond G3, the lines through types G to M have been left much as in the original diagram. As will be seen later, there is some evidence favoring the reality of the minimum near K0. If the reductions to bolometric magnitude given on page 194 be applied, the variation of mass with spectrum becomes fairly

TABLE XVI

MASSES AND DENSITIES OF CEPHEID VARIABLES

STAR	SPECTRUM		P	CALCULATED				CALCULATED *minus* FIG. 3					
	Est.	H.L.		M	$\log \rho$	$\mathcal{M}$		$\log \rho$		$\log \mathcal{M}$		M	
α Ursae Minoris...	F9	....	3.97	−1.8	−3.10	5.0		+0.01		+0.03		0.0	
TU Cassiopeiae....	....	F4.5	2.14	−1.2	−2.56		5.5		+0.08		+0.11		−0.1
SU Cassiopeiae....	F4	F1.5	1.95	−1.2	−2.48	3.0	2.9	−0.17	−0.09	−0.22	−0.24	+0.4	+0.3
SZ Tauri..........	F8p	F6	3.15	−1.6	−2.90	4.6	4.4	+0.01	+0.09	0.00	−0.01	0.0	−0.3
RX Aurigae.......	F9p	....	11.63	−3.4	−4.04	5.2		−0.18		−0.21		+0.3	
T Monocerotis.....	G1p	F8.5	27.01	−4.7	−4.76	13.8	13.2	0.00	+0.03	+0.06	+0.04	0.0	−0.1
RT Aurigae.......	F8p	F5	3.73	−1.8	−3.04	5.0	4.8	−0.03	−0.05	+0.02	−0.01	0.0	+0.1
W Geminorum....	G0p	F6.5	7.92	−2.8	−3.70	5.8	5.5	+0.01	−0.06	−0.07	−0.09	0.0	+0.1
RS Boötis.........	....	A4	0.38	−0.3	−1.06		9.6		+0.18		+0.17		−0.2
ζ Geminorum......	G0p	....	10.15	−3.2	−3.92	6.0		−0.06		−0.12		0.0	
X Sagittarii.......	F9p	F5	7.01	−2.6	−3.60	4.8	4.6	−0.14	−0.21	−0.12	−0.15	+0.3	+0.5
Y Ophiuchi.......	....	F9	17.11	−4.0	−4.36		14.4		+0.13		+0.16		−0.2
W Sagittarii.......	G0p	F6	7.60	−2.7	−3.66	5.5	5.2	0.00	−0.12	−0.07	−0.10	0.0	+0.3
Y Sagittarii.......	G2p	F9	5.77	−2.3	−3.42	9.6	9.1	+0.24	+0.27	+0.24	+0.22	−0.5	−0.5
U Sagittarii.......	G0p	....	6.74	−2.5	−3.56	6.0		0.00		0.00		0.0	
YZ Sagittarii......	G1p	....	9.55	−3.1	−3.86	9.1		+0.05		+0.08		−0.2	
TT Aquilae.......	G2p	....	13.75	−3.6	−4.18	11.5		+0.13		+0.11		−0.3	
RR Lyrae.........	F0p	A5.5	0.57	−0.4	−1.42	6.6	6.3	+0.09	+0.02	+0.09	+0.03	−0.2	−0.1
U Aquilae.........	G0p	F9	7.02	−2.6	−3.60	5.5	5.2	+0.01	+0.19	−0.05	−0.07	0.0	−0.4
XZ Cygni.........	....	A3	0.47	−0.3	−1.24		5.5		−0.10		−0.08		+0.2
U Vulpeculae......	G0p	F7	7.99	−2.8	−3.70	5.8	5.5	+0.01	−0.01	−0.07	−0.09	0.0	0.0
SU Cygni.........	F7p	F4	3.85	−1.8	−3.08	3.5	3.3	−0.17	−0.19	−0.16	−0.18	+0.3	+0.4
η Aquilae.........	F9p	F7	7.18	−2.6	−3.62	4.6	4.4	−0.11	−0.03	−0.14	−0.15	+0.3	+0.1
S Sagittae.........	G1p	F8.5	8.38	−2.9	−3.74	7.9	7.6	+0.12	+0.15	+0.06	+0.04	−0.2	−0.3
X Cygni..........	G3p	....	16.38	−3.9	−4.32	19.0		+0.24		+0.28		−0.5	
T Vulpeculae......	F9	F5	4.44	−2.0	−3.20	5.2	5.0	+0.01	−0.11	+0.02	−0.02	0.0	+0.2
δ Cephei..........	G0	F6	5.37	−2.2	−3.36	6.3	6.0	+0.05	−0.07	+0.08	+0.05	−0.1	+0.1
W Serpentis.......	F9p	....	14.15	−3.7	−4.20	6.3		−0.19		−0.16		+0.4	
Average difference	....	....	....	....	....	....		±0.08^5	±0.11	±0.10	±0.10	±0.17	±0.22

regular, the course of the topmost line ($\mathcal{M}=10$) being indicated by the dotted curve in Figure 3.

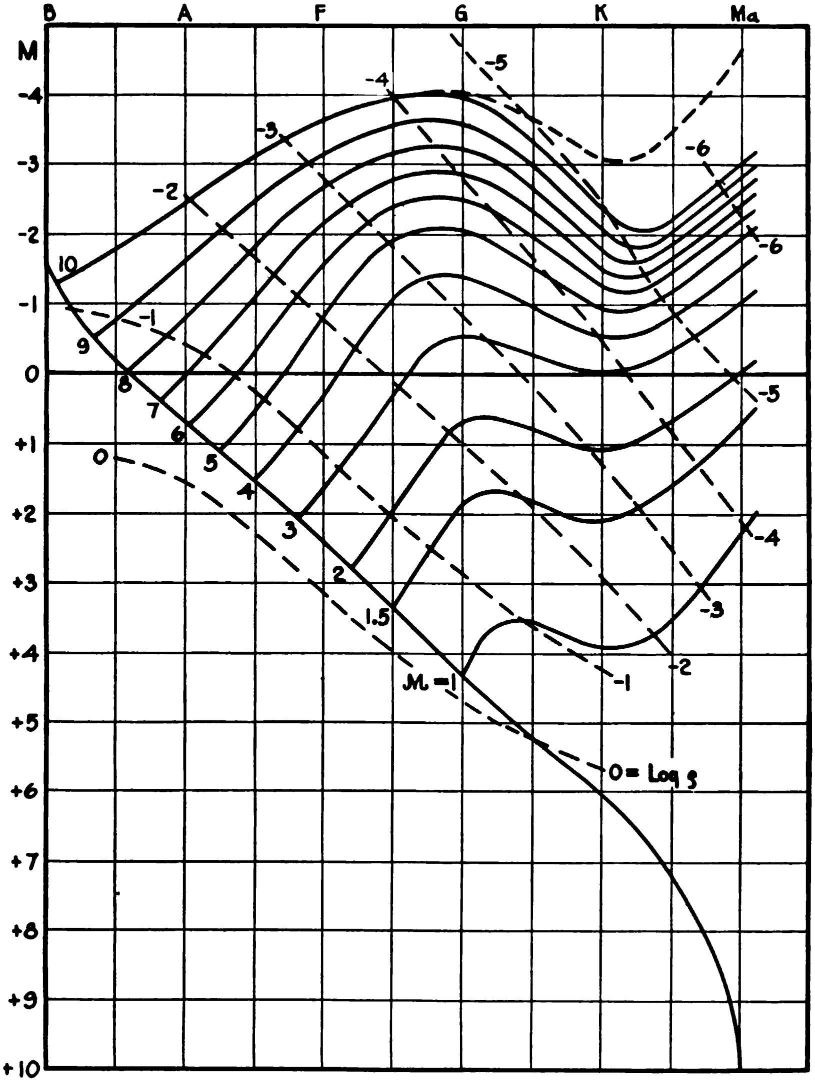

FIG. 3.—Distribution of mass and mean density, revised with the aid of Cepheid variables. The dotted curve at the top indicates the reduction to bolometric absolute magnitude for late-type stars of mass 10.

The curves of equal density are little changed. They still remain sensibly linear, but are no longer so closely parallel. Formula (23) has not wholly lost its applicability, for it still expresses approximately the relation of density to spectrum and absolute magnitude, though less accurately than before.

It will be noted that the densities along the dwarf branch are consistent with the values demanded by the density-distribution defined by the lines log $\rho=0$, -1, -2, etc. In other words, if we interpolate, say, for the value log $\rho=-0.4$ ($\rho=0.4$) along the line $\mathcal{M}=6$, we find the point where it belongs, namely, on the dwarf branch. This detail is of some importance because it shows that the constants in the formulae for log ρ and log $\mathcal{M}$ have been correctly determined. The spacing of the lines is fixed by the periods of the variables; their position as a group, by the condition that they must be consistent with the densities adopted for the dwarfs, whose values are independent of any assumption as to equipartition.

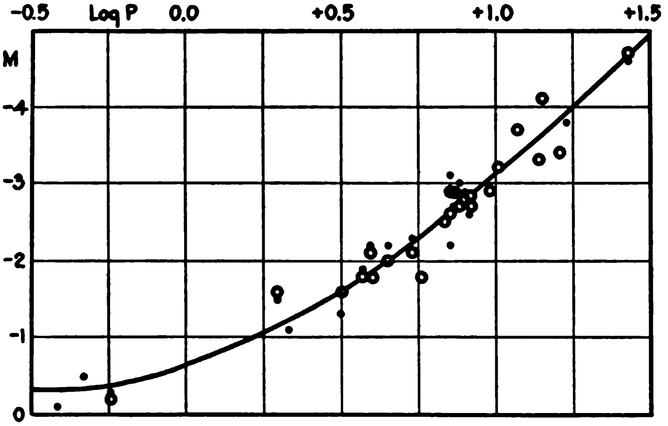

Fig. 4.—Absolute magnitudes of Cepheids interpolated from Fig. 3 with spectral type and log ρ from equation (24) as arguments. Circles, "estimated" spectra by Adams; points, hydrogen-line spectra by Shapley. The curve is Shapley's period-luminosity relation.

Presumably the diagram in Figure 3 is to be preferred to that in Figure 2. Certainly that is the case when Cepheid variables are concerned; and it seems likely that this holds also for the stars at large. The mass lines in Figure 2 pass *through* the points defined by the energy-constant and the velocities; no allowance for observational uncertainty by smoothing between the types was attempted; and with the exception of the terminal points on the dwarf branch everything preceding F5 is an extrapolation. On the other hand, an assumption of comparability is here involved, and it is perhaps worth remarking that if we reject the original diagram, we may thereby exclude relations between mass, density, luminosity, temperature, etc., which differentiate stars of constant light from Cepheid variables.

In view of the tentative character of the results, a detailed tabulation of the mass and density relations is unnecessary; Figure 3 will answer all practical purposes to which the results can be adapted. The data for the dwarf branch given in the first part of Table XIV, which are the most reliable, have already been referred to. To these have been added, by way of numerical illustration, results for the giants of zero absolute magnitude and for those whose masses are equal to 10.[1]

As a further illustration, the masses and densities corresponding to the spectra and absolute magnitudes of the individual stars measured by Pease with the interferometer have been given in the last lines of Table XIII. As a source of error, the values of M for these stars are probably as serious in respect to mass and density as the uncertainties of Figure 3 itself. Equation (22) shows that an error of 0.5 in M means an uncertainty factor of 2 in the density, whereas it seems unlikely that values from the diagram should be in error by so large an amount. In the case of an individual star, dispersion may of course enter, since the diagram gives mean values; but the probability of the occurrence of large deviations from the mean is small.

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[1] Since the displacement of spectral lines required by relativity is proportional to $\mathcal{M}/R$ (R=radius of star) the results in Table XIV have a direct bearing on this question. The values of $\mathcal{M}/R$ do not run parallel to those of the K-term. Hence this term cannot be wholly identified with the relativity displacement, although the latter may of course be involved. As an illustration, the K-term for the B stars is about four times that for the A's. The ratio of the displacements on the other hand would be about 1.3. The most extreme deviation, however, is found by comparing the B's with the giant M stars. Their K-terms are of the same order of magnitude, whereas the red-displacement of the lines in M stars should be of the order of 0.01 that of the B's. See footnote No. 2, p. 191.

13. ON DISPERSION IN THE MASSES OF CEPHEIDS

From Shapley's work it appears that, for the variables in globular clusters at least, the deviations in M from the values determined by the period-luminosity relation are of the same order as the observational errors in the apparent magnitudes,[1] which of course must be used for any comparison. Practically, therefore, M is uniquely determined by P. Hence, by (26), the dispersion in the mass is related to that arising from such differences in j as may be associated with the same value of P, and it is an observational fact that the latter are small.

Direct evidence is also afforded by the comparison of the values of $\mathcal{M}$ for the Cepheids calculated by (26) with those interpolated from Figure 3. The average difference in log $\mathcal{M}$ is ± 0.10, corresponding to about 25 per cent in $\mathcal{M}$. But what part of this may be attributed to uncertainty in the spectrum?

With proper allowance for the systematic difference of 0.3 between the hydrogen line and estimated spectra, the internal agreement is such as to indicate an average uncertainty of 0.05 of an interval. Assume an adopted value of the type to be too late by this amount. From (26)

$$\text{Mod.}\ \frac{d\mathcal{M}}{\mathcal{M}} = +0.6\ dj. \qquad (27)$$

For the region F0–G0, j increases about 0.07 for the assumed increase in the spectrum. Hence, the calculated $\mathcal{M}$ will be 10 per cent too large. On the other hand, we find from Figure 3, for F5, $M = -2.5$, as an average, that the same error in the spectrum leads to an interpolated $\mathcal{M}$ about 3 per cent too small. The resulting errors are in opposite directions; whence, in the difference, the uncertainty becomes 13 per cent.

This average percentage error, combined with the dispersion and the errors of interpolation, must equal the average deviation of 25 per cent from the mean masses of the diagram. Neglecting the interpolation errors, whose inclusion would make the dispersion still smaller, we find the probable dispersion $r(\mathcal{M})=19$ per cent. As far as the evidence goes, we should therefore expect one-half the Cepheids to have masses lying between 0.8 and 1.2 times the mean mass. Since the Cepheids are a very special class of objects, it would be too much to assume that this result holds for all classes of stars (see Section 7).

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[1] *Mt. Wilson Contr.*, No. 151, p. 26; *Astrophysical Journal*, **48**, 114, 1918.

14. MASSES AND DENSITIES OF CEPHEID VARIABLES

The calculated values of the masses and densities used for the comparisons with the diagram of Figure 3 have been collected in the sixth and seventh columns of Table XVI, while their deviations from the adopted distribution are in the following columns. The formulae used are:

$$\log \rho = -2 \log P - 1.90 \tag{29}$$

$$\log \mathcal{M} = -2 \log P - 0.6\,(M-j) + 0.87 \tag{30}$$

in which the constants are the means for the two sets of data, the individual values differing by only 0.02. In case the spectral type is the median value derived from the hydrogen lines, the median value of M, given directly by the period luminosity relation, is to be used. For estimated spectra, the minimum M is required, although it is simpler to subtract 0.3 from the estimated spectrum and use the median M.

The values of both density and mass depend upon the assumed constancy of the numerical term in (29), but this assumption apparently leads to results of the right order, although for stars of short period the percentage deviation may be considerable.

For individual stars the calculated values are probably to be preferred to those interpolated from Figure 3. This follows not merely because of the dispersion in mass, but also because the differences for stars having determinations based on both estimated and hydrogen-line spectra are almost always of the same sign. This indicates real deviations from the mean distribution, in addition to those arising from the unavoidable errors of spectral classification. In such cases the values from (29) and (30) are likely to be nearer the truth than those derived from the diagram.

The results in Table XVI, both calculated and interpolated, may be compared with the masses and densities derived by Eddington[1] for certain Cepheids on the basis of his theory of radiative equilibrium. The agreement shown by Table XVII is remarkable

TABLE XVII

COMPARISON WITH EDDINGTON'S RESULTS FOR CEPHEID VARIABLES

STAR	MASS			MEAN DENSITY		
	Eddington	Eq. (30)	Fig. 3	Eddington	Eq. (29)	Fig. 3
Y Ophiuchi	13.0	14.4	10	0.000080	0.000044	0.000032
ζ Geminorum	7.4	6.0	7.9	.00046	.00012	.00014
S Sagittae	6.1	7.8	6.9	.00019	.00018	.00013
W Sagittarii	5.5	5.4	6.5	.00041	.00022	.00025
η Aquilae	5.2	4.5	6.2	.00033	.00024	.00028
X Sagittarii	5.1	4.7	6.4	.00025	.00025	.00038
Y Sagittarii	4.3	9.4	5.5	.00027	.00038	.00021
δ Cephei	4.0	6.2	5.3	.00052	.00044	.00045
T Vulpeculae	3.6	5.1	5.1	.00072	.00063	.00071
SU Cygni	3.1	3.4	5.0	.00160	.00083	.0013
RT Aurigae	3.0	4.9	4.8	.00110	.00091	.0010
SZ Tauri	2.8	4.5	4.5	.0015	.0013	.0011
SU Cassiopeiae	2.3	3.0	5.0	.0028	.0033	.0045
RR Lyrae	1.7	6.4	5.6	0.015	0.038	0.033

when it is considered that the two methods approach the problem from entirely different directions. There are a few cases of serious discordance, and some systematic divergence, but nothing so large as might have been expected. The agreement with (29) has been commented upon by Eddington himself.

The geometrical mean mass of the twenty-four Cepheids in Table XVI is 6.2, which happens to agree almost exactly with that derived from the principle of equipartition. This can scarcely be more than accidental, however, for the velocities of only fifteen of these stars are available.

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[1] *Monthly Notices*, **79**, 5, 1918.

15. AVERAGE HEAT CONTENT AND CEPHEID VARIATION

On the basis of the theory of radiative equilibrium Eddington[1] finds for the central temperature of a star

$$T_c \propto \rho^{\frac{1}{3}} \mathcal{M}^{\frac{2}{3}} \beta m \tag{31}$$

where ρ is the mean density, β the ratio of radiation pressure to gas pressure, and m the atomic weight which is here assumed to be constant. The formula may also be written

$$T_c \propto \frac{\beta \mathcal{M}}{R} \tag{32}$$

R being the radius. $R=D/2$ is given by (18) or an equivalent expression; β is a function of $\mathcal{M}$, and $\mathcal{M}$ in turn appears in his formulae as a function of the luminosity alone, which can be obtained from (25). T_c can therefore be calculated for any star. For thirteen Cepheids with periods ranging from 2 to 17 days Eddington[2] finds that T_c/β is sensibly constant. There are accidental deviations, but no progressive change. Moreover, for two theoretical stars with periods of 4.5 and 30.8 days the values of T_c/β are practically identical. This suggests as a possible condition for Cepheid variation

$$\frac{\mathcal{M}}{R} = \text{const., or } \rho \mathcal{M}^2 = \text{const.} \tag{33}$$

Since $\mathcal{M}^2/R$ is proportional to the total heat-content of a star,[3] the physical significance of (33) is that the average heat-content per unit mass is a constant, or, in another form, that the gravitational potential at the surface of the star is constant for the Cepheids. Shapley[4] has discussed this condition from the standpoint of the relation of period to absolute magnitude and finds agreement with the observed period-luminosity relation (25). We may proceed somewhat differently by combining (24) and (26) in accordance with (33). This gives the theoretical period-luminosity relation

$$M = -5 \log P + j + \text{const.} \tag{34}$$

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[1] *Monthly Notices*, 77, 601, 1917.

[2] *Ibid.*, 79, 5, 1918; 181, 1919.

[3] Jeans, *Problems in Cosmogony*, p. 191, 1919.

[4] *Mt. Wilson Contr.*, No. 190, p. 7; *Astrophysical Journal*, 52, 79, 1920.

A comparison of the values of M from (34) for the twenty-eight stars in Table XVI with those from the observed relation, (25) above, gives results which are shown graphically in Figure 5. The agreement is not good, the divergence amounting to about a magnitude between $\log P = 0.25$ and 1.5, with even larger differences for the cluster-type Cepheids, which are not shown in the diagram.

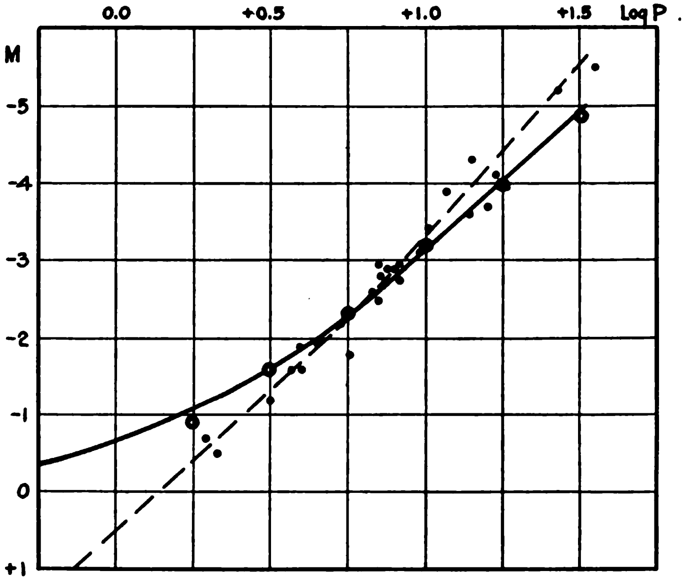

FIG. 5.—Absolute magnitudes of Cepheids based on the assumption of constant average heat content. The points are values for individual stars from equation (34). The circles, from equation (36), show the agreement with Shapley's period-luminosity relation when the term involving the specific heats in Eddington's expression for the density is included.

It will be noted that (33) is an empirical result which follows from an application to Cepheids of the period-luminosity relation (25) and the general theory of radiative equilibrium, but its derivation does not in any way involve the theory of Cepheid variation itself. If valid, it ought, however, in combination with the latter theory, to lead back to relation (25). But the comparison, as made, does not give complete agreement. Since the theory of Cepheids has entered only through (24), the difficulty presumably lies with that equation, which has been assumed to be applicable. If instead we use the corresponding relation derived by Eddington[1] himself, the difficulty disappears. For (24) we must substitute

$$\log \rho = -2 \log P - 2 \log (\gamma a)^{\frac{1}{2}} + \text{const.} \qquad (35)$$

where $(\gamma a)^{\frac{1}{2}}$ is a function of β and of the ratio of the specific heats, and changes slowly with the mass. With this emendation, (34) becomes

$$M = -5 \log P + j - 5 \log (\gamma a)^{\frac{1}{2}} + \text{const.} \qquad (36)$$

The factor depending on the specific heats enters to the fifth power and thus becomes important. Its neglect in (34) causes the divergence shown by Figure 5.

The necessary corrections are easily calculated, however, with the aid of Table V of Eddington's paper. The values of $1-\beta$ required for the interpolation of $(\gamma a)^{\frac{1}{2}}$ are given in *M.N.*, **77**, 602, 1917, under the heading "Molecular Weight 2," with the mass as argument. For $\mathcal{M}$ we may use the values in Table XVI, or more consistently with the present procedure, we may calculate $\mathcal{M}$ from

$$\log \mathcal{M} = -0.2\,(M - j) + \text{const.} \qquad (37)$$

which follows at once from (18) and the condition $\mathcal{M} \propto R$ given by (33). The latter method was adopted, the constant in (37) being determined so that the mean $\mathcal{M}$ from (37) agrees with that from the diagram in Figure 3. We are interested only in the variation of $(\gamma a)^{\frac{1}{2}}$, which is practically the same for all possible values of the ratio of specific heats. $\Gamma = 1\frac{4}{9}$ was used, however, since Eddington's results show that this agrees well with the data of observation. With the constant in (36) equal to 1.48, the values of $-5 \log (\gamma a)^{\frac{1}{2}}$ are as shown in the third column of Table XVIII. The sums of the remaining terms on the right of (36) are given in the second column under the heading M_0. These are the co-ordinates of the dotted curve in Figure 5 and were read directly from a large-scale diagram. The calculated values of M from (36) (fourth column) when compared with those from (25) give the residuals in the last column of the table. For values of P between 3 and 30 days the agreement is very close. For $P<2$ days there is still a large divergence, but that is to be expected. For all but the cluster-type Cepheids, therefore, the theoretical relation (36) is practically the equivalent of the observed period-luminosity relation (25).

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[1] *Monthly Notices*, **77**, 15, 1918.

TABLE XVIII

THEORETICAL AND OBSERVED PERIOD-LUMINOSITY RELATION

Log P	M_0	$-5 \log (\gamma a)^{\frac{1}{2}}$	M (36)	M (25)	O−C
0.25	−0.45	−0.45	−0.90	−1.08	−0.18
0.50	−1.30	−0.29	−1.59	−1.58	+0.01
0.75	−2.25	−0.08	−2.33	−2.26	+0.07
1.00	−3.34	+0.15	−3.19	−3.15	+0.04
1.25	−4.43	+0.45	−3.98	−4.05	−0.07
1.50	−5.54	+0.68	−4.86	−4.95	−0.09

Shapley's discussion referred to above starts with the assumption of constant heat-content and leads to a theoretical relation which agrees well with (25), but does not involve Eddington's theory of Cepheids because his comparison is based on equation (24) instead of (35). The close agreement found by Shapley is due partly to the fact that his final formula neglects $2.5 \log (\gamma a)^{\frac{1}{2}}$ instead of $5 \log (\gamma a)^{\frac{1}{2}}$, which by itself would lead to a divergence from (25) equal to one-half that shown in Figure 5, and partly to his readjustment of the mean atomic weight.

The matter is of interest from two or three points of view: First, because it shows the consistency of Eddington's theory of Cepheid variation with the general theory of radiative equilibrium, in that it is necessary to include the term depending on the specific heats in order to work back to the observed period-luminosity relation which was one of the premises; and second, because it exhibits the empirical result (33) in another form and possibly gives weight to the suggestion that constant average heat-content may be a determining circumstance in Cepheid variation.

The fact that the term which has been under discussion enters to so high a power subjects Eddington's theory of Cepheid variation to a severe test. Something of the sort is also true of the empirical condition (33). Assume for a moment that the relation $\mathcal{M} \propto R$ is not exact, and suppose that all the variations which occur are thrown into the mass. The maximum systematic deviation in log $\mathcal{M}$, by (18), (22), and (37), will then be equal to 0.4 of that shown by the differences in the last column of Table XVIII, which is well within the uncertainty of the comparison.

This close agreement justifies the mention of other relations which follow at once from the assumption of constant average heat-content, combined with the following equations of radiative equilibrium:[1]

$$\left.\begin{aligned} T_e^4 &\propto g(1-\beta) \propto \mathcal{M}^{\frac{1}{3}}\rho^{\frac{2}{3}}(1-\beta) \\ T_c &\propto \mathcal{M}^{\frac{2}{3}}\rho^{\frac{1}{3}}\beta \\ 1-\beta &\propto \mathcal{M}^2\beta^4 \end{aligned}\right\} \qquad (38)$$

where T_e and T_c are the effective and central temperatures, respectively, g the acceleration due to gravity, and $1-\beta$ the ratio of radiation pressure to gravitational attraction. Assuming $\mathcal{M} \propto R$ we find then

$$\left.\begin{aligned} &\mathcal{M}^2\rho = c_1;\ \mathcal{M}g = c_2;\ Rg \propto c_3;\ \rho \propto g^2 \\ &T_c \propto \beta;\ T_e \propto T_c\mathcal{M}^{\frac{1}{4}} \propto \mathcal{M}^{\frac{1}{4}}\beta \propto T_c\rho^{-\frac{1}{8}} \end{aligned}\right\} \qquad (39)$$

where c_1, c_2, and c_3 are constants. Further, with the aid of (35)

$$\mathcal{M} \propto (\gamma a)^{\frac{1}{2}}P;\ T_e \propto T_c(\gamma a)^{\frac{1}{8}}P^{\frac{1}{4}} \propto \beta(\gamma a)^{\frac{1}{8}}P^{\frac{1}{4}}. \qquad (40)$$

The relation $T_e \propto \mathcal{M}^{\frac{1}{4}}\beta$ is of special interest. Since β diminishes faster than $\mathcal{M}^{\frac{1}{4}}$ increases, T_e must decrease with increasing $\mathcal{M}$. The change for the range in $\mathcal{M}$ shown above corresponds to an increase of about one spectral interval, which agrees well with the observed range in spectral type. At the same time M decreases from -1.2 to -5.1.

The formulae of radiative equilibrium admit of no dispersion in the case of individual stars. This is also true of the relations (39) and (40) for the Cepheids. In the case of the masses and densities, we have already found reasons for believing that this is not strictly in accordance with the facts. Although the dispersion is small, it is in part undoubtedly real, for at present the theory necessarily neglects the modifications of gravitational acceleration produced by the rotation of the stars on their axes, which must influence the relation of radiation to other stellar characteristics.

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[1] Eddington, *Monthly Notices*, 79, 180, 1919.

16. RELATION TO RUSSELL'S FREQUENCY DIAGRAM—REMARKS ON THE DISTRIBUTION OF MASS AND DENSITY

The justification for the discussion of Eddington's theory of Cepheids in this place is the light thrown upon the distribution of mass and density. Equations (35) and (37), for example, afford a basis for a further revision of the mass- and density-curves; but an additional approximation seems scarcely justified at present. Table XVII shows already a surprisingly good agreement between his results and those in Figure 3. Further modification would affect mainly the stars of masses 10 or larger, and these are not numerous. Moreover, Eddington himself has repeatedly emphasized the fact that his theory is but a first approximation, in that certain simplifying assumptions had to be introduced in order to make a beginning. Figure 3 is therefore allowed to stand as it is for the present, although the data in Table XIX are added to show, in a general way, what would result, for the Cepheids at least, from rigorous application of the theory. The values of M and the spectrum may be regarded as co-ordinates for the location of corresponding values of $\mathcal{M}$ in a diagram similar to Figure 3.

TABLE XIX

Mass, Absolute Magnitude, and Spectrum, Eddington's Theory of Cepheids

$\mathcal{M}$	M	Sp.	$\mathcal{M}$	M	Sp.
2	−0.9	F1.0	10	−3.4	F8.0
3	−1.4	F4.0	12	−3.6	F8.4
4	−1.9	F5.2	15	−4.0	F9.0
6	−2.5	F6.5	20	−4.6	F9.3
8	−3.0	F7.5	25	−5.1	F9.6

The values of the masses are adjusted to give a mean value equal to that derived from Figure 3.

It is, of course, not to be supposed that the stars are uniformly distributed along the mass-curves of Figure 3. In general they cluster about certain lines of maximum frequency which, with one exception, bear little relation to the lines of equal mass. The most conspicuous of the frequency lines is that defined by the dwarfs, which continues through the A's and joins smoothly with the line of maximum frequency of the B stars. Along this line we have found a regular decrease in the average mass with increasing spectral type and absolute magnitude.

The number of A stars of very high luminosity seems to be relatively small.[1] If this is really the case, there must be a region of low frequency in the diagram between the most luminous B's and the very bright giants of the later types. Beyond F0 the giants fall into two groups, the Cepheids and pseudo-Cepheids, and a large group of G, K, and M stars whose absolute magnitudes lie mainly between $M=-0.5$ and $+2.5$. The Cepheids and pseudo-Cepheids attain very high luminosities and show a rather definite correlation line running approximately from $M=-0.4$, F4, to $M=-4$, G8, which in general cuts the mass lines obliquely. For the Cepheids alone, its course is determined by the intersections of the mass lines of Figure 3 with those defined by equation (37), which, empirically at least, is a condition of Cepheid variability.

The frequency line for the second and more numerous group of giants seems, on the other hand, to be definitely related to the distribution of mass. It has a clearly marked minimum at K0 and coincides approximately[2] with the mass line $\mathcal{M}=2$ as shown in Figure 3. The chief difference is a steeper descent from G0 to K0 and an even more pronounced minimum than is shown by the mass line. The coincidence is probably significant, for Eddington[3] finds on theoretical grounds that, among the giants, masses of this order of magnitude may be expected to have the highest frequency.

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[1] The frequency diagram of Russell for the helium stars of Kapteyn and the stars in the list of spectroscopic parallaxes is illustrated in Annual Report, Mt. Wilson Observatory, *Year Book Carnegie Institution of Washington*, 1921.

[2] The position of this correlation line and of the line of maximum frequency for the G, K, and M giants is shown in Fig. 1. This figure also shows the course of the equal-mass line $\mathcal{M}=2$. The correlation of spectral type with absolute magnitude for the most luminous stars of our own immediate system is essentially that found by Shapley for the brightest stars in globular clusters; and it now appears from results not yet published that in some of the galactic clouds a similar correlation with apparent magnitude is to be found. It should be possible in such cases to estimate roughly the distances of the clouds. Provisional results indicate values of the order of 20,000 to 50,000 light years. See *Annual Report, loc. cit.*

[3] *Report of the British Association for the Advancement of Science*, 1920, p. 43.

These circumstances seem to afford justification for the peculiar course of the mass lines for the late-type giants, which otherwise must depend solely upon the applicability of the principle of equipartition.

The minimum itself, or what remains of it after the reduction of the visual absolute magnitudes to bolometric values in the manner indicated above, must not be taken too seriously, however, for the residual irregularity in the mass lines is little in excess of the small systematic uncertainties affecting the absolute magnitudes of the different spectral types.[1]

In this connection the result of page 183 for the masses of fourteen giant visual binaries may be recalled. Divided by 1.75 to reduce to the central component, the mean mass becomes 1.7 ± 0.9. For the same stars we find by interpolation from Figure 3, with spectral type and absolute magnitude as arguments, the mean value $\mathcal{M} = 2.7 \pm 0.2$ The agreement is not good, but falls within the limits of uncertainty incident upon the small number of stars available for the comparison.

The decrease in mass along the dwarf branch from $\mathcal{M} = 10$ at Bo to $\mathcal{M} = 0.6$ at Ma raises a question of much interest, especially when it is recalled that the dispersion in mass is certainly small.[2] Practically, large masses are not to be found in our lists of dwarf stars. And yet the large-mass early-type stars are numerous. Why, therefore, do we not find more large masses among the dwarfs?

Several possibilities must be considered. Thus we might assume that the stars now on the dwarf branch (including the B's, as usual) began their development at much the same time, and that those of largest mass have only partially run their course and are now in the B stage. The assumption would seem less arbitrary could we suppose that all the stars began their evolutionary careers at the same time; but this apparently is not the case, for numerous objects with only moderate masses of 2 or 3 are still in an early stage of development. The most serious objection to this explanation, however, is that it takes no account of a peculiar kind of selection.

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[1] Strömberg, *Mt. Wilson Contr.*, No. 220; *Astrophysical Journal*, 55, 11, 1922.

[2] Russell has also discussed this correlation., *Observatory*, 37, 173, 1914.

Let us therefore make the contrary assumption and suppose that stellar development has continued so long that all possible masses really occur among the dwarfs of late type. Among these, the large masses, say of the order of 10, will be of very low frequency as compared with the modal value for all types together. Kapteyn and van Rhijn place the maximum of the frequency-curve of absolute magnitudes at $M = +7.7$. The modal value of the mass therefore should be about 0.7, and since the dispersion is small, the large masses must be very infrequent.

Consider now only the stars in our lists of data. The dwarfs among these will all be included in a restricted region of space near the sun. The small masses will greatly predominate and the mean mass will be low. The giants, on the other hand, will be scattered through a large volume of space, and their average mass will be large, because only stars of large mass can attain high luminosity. The B's among them will form the collection at the upper end of the dwarf branch whose mean mass we have found to be 10. The total number of these in our catalogues will be considerable, because they represent those present in a large volume, whereas the large masses among the dwarfs will be absolutely as well as relatively infrequent, because we can see only those which are near us.

The stars of intermediate luminosity fill in the gap, and we must therefore expect to find a correlation of mass with absolute magnitude and spectral type very similar to that actually observed. The average mass decreases with advancing type; at the same time the number of stars of large mass in our catalogues decreases rapidly and the smaller masses appear in succession and in increasing numbers because they can attain the temperatures corresponding to the later spectral types.

One other possibility should also be borne in mind, namely, a decrease in mass through loss of energy by radiation. From the standpoint of relativity[1] any change dE in the total energy of a system implies a corresponding change of dE/c^2 in the inertial mass, where c is the velocity of light. A similar conclusion follows also on the basis of the Newtonian mechanics[1] from the phenomenon of radiation pressure, for, unless we attribute mass to radiation, there are certain cases, at least, in which neither the energy nor the motion of the center of gravity will be conserved. Since the principles of mechanics are inductions, we cannot be sure that they apply to physical phenomena with the degree of precision implied by assuming that mass actually changes with changes in the energy. But considerations of consistency are sufficient to give the matter importance, and it derives added interest from the fact that a decrease in mass has been invoked as a means of accounting for the enormous unknown supply of energy which is now generally recognized as an essential feature of stellar phenomena. The suggestion that the packing of electrons and the nuclei of hydrogen atoms to form nuclei of heavier elements, with an accompanying loss in mass which reappears in the form of great quantities of radiant energy, attaches itself naturally to the relation between mass and energy that harmonizes with fundamental mechanical principles, whether of Einstein or of Newton.

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[1] Einstein, *Annalen der Physik*, **18**, 639, 1905.

Of these three factors which may affect the correlation of mass and spectral type, selection certainly enters in the manner indicated, but its quantitative evaluation is at present impossible. This leaves the operation of the other two factors entirely speculative. If, however, there is any appreciable change in the mass of the stars during their development, they will not follow the mass lines of Figure 3, but cut across them obliquely; and this leads to a peculiarity of these lines that immediately attracts attention.

The equal-mass lines do not meet the dwarf branch tangentially, as one might expect on the basis of a simple gravitational contraction with its accompanying changes of temperature, but intersect the frequency line at large angles. A comparison of Figures 2 and 3 shows that in this particular there is a large element of uncertainty; and yet the evidence seems to exclude the possibility of tangential intersections, unless we are prepared to abandon the principle of equipartition at the point where it seems to be best justified, namely, in the vicinity of the dwarf branch.

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[1] *Ibid.*, 20, 627, 1906. This result seems also to have been pointed out by others--Planck, for one, I believe—although I have been unable to locate the source.

The space-velocities of stars of any given type, say F5, show a definite correlation with absolute magnitude, the mean velocity increasing continuously as the brightness decreases. If equipartition holds for the stars of a given type which are near the frequency line of the dwarfs, as it does in the mean for the stars of different types scattered along the frequency line, there must be a continuous decrease in mass downward along the vertical lines of Figure 3. The data of Table X give a numerical expression to this relationship of mass to luminosity, which, quantitatively, should be most exact for values of M near the frequency line of the dwarfs. Now on this line, for spectral type F5, as an illustration, we find a mean mass of 1.5. Above this point in the diagram, that is, for more luminous stars, the masses are larger; below they are smaller. If the decrease in mass downward is really continuous, it follows at once, since the mass also decreases continuously along the dwarf frequency line, that curves connecting equal masses cannot meet the frequency line tangentially, but must intersect it at a considerable angle. In fact Figure 2 shows that the strict application of the principle of equipartition leads to angles in excess of 90°, counting as 0° the tangential junction required by gravitational contraction. The evolutionary path of any given star, however, when once it has joined the dwarf branch, must lie in the general direction of the frequency line.

The explanation of the contradiction lies in the peculiar selection which determines the decrease in mass along the dwarf branch. Consider for example two dwarf stars of Type F5, one a large mass which in its development has followed the dwarf branch downward from some earlier type, the other a star of mass so small that it has been unable to attain the high temperatures of the very early types and joins the dwarf branch at Type F5. The star of large mass will be the brighter of the two. Hence the path of the small star must intersect the line of maximum frequency at some point a little in advance of F5 in order that it may swing into position below the larger star. The path of the larger star, on the other hand, will lie above the frequency line. Similar reasoning applies to stars of intermediate mass and it is evident that the entire group of F5 dwarfs will show a correlation of mass with absolute magnitude similar to that found from the principle of equipartition. In their further evolution these stars will move as a group along paths approximately parallel to the line of maximum frequency. But these paths do not and cannot coincide with the equal mass lines of Figure 3, for when the group in question is at F5, there is at G0 another group, whose luminosities show a similar correlation with mass, but whose masses, because of selection, are systematically smaller than those at F5. If the points in the diagram corresponding to equal masses in these two simultaneously observed groups be connected, the lines of equal mass will cut the frequency line at high angles as shown in Figure 3. Hence there is no conflict between the mass-distribution shown by this diagram and the gravitational theory of stellar development. The only point is that in the vicinity of the dwarf branch the stars do not follow the equal mass lines.

17. IONIZATION AND MEAN ATOMIC WEIGHT

The final equations of Eddington's theory[1] are

$$L=\frac{4\pi cG\mathcal{M}}{k}(1-\beta) \tag{41}$$

$$1-\beta=0.0026\ \mathcal{M}^2\beta^4m^4 \tag{42}$$

where c and G are constants. These formulae refer to the giant stage and assume that the mean atomic weight is independent of the temperature. Variations in m during the star's development would affect the relation between luminosity and spectral type. The absolute value of m fixes the relation between mass and radiation-pressure β, and hence determines the values of the mass which are most likely to occur.

It is unnecessary to repeat here Eddington's arguments in favor of a small value of m. In general these seem to be only strengthened by the results of recent investigations of ionization phenomena. Eggert,[2] for example, has considered the behavior of the iron atom when subjected to pressures and temperatures comparable with those at the center of a typical giant star and finds that 16 of the

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[1] *Astrophysical Journal*, 48, 208, 210, 1918.

[2] *Physikalische Zeitschrift*, 20, 570, 1919.

26 electrons would be stripped from the atom, thus reducing the mean "atomic" weight to 3.3.

It is easy to extend this result to other parts of the star. Using the reaction isobar of Nernst and basing the calculation of the energy of dissociation on the dynamical relations of the Bohr atom, Eggert finds as the condition for the removal of 8 electrons from 99 per cent of the atoms[1]

$$2.68+0.4 \log P=\frac{-2\times 10^5}{T_i}+\log T_i \tag{43}$$

in which the pressure P is measured in atmospheres. For the separation of 16 electrons the corresponding condition is approximately the same as (43) except for the appearance of an additional factor of 10 in the first term on the right.

Equation (43) and its alternate may be used to calculate the values of T_i corresponding to the pressures at different points along the radius. The comparison of the ionization temperatures thus found with the corresponding stellar temperatures should then give some indication as to the variation in ionization throughout the star.

The total pressure P is given by the ordinary gas equation, divided by β, the factor arising from radiation-pressure.

$$P=\frac{p}{\beta}=\frac{\mathfrak{R}}{\beta m}\rho T \tag{44}$$

T and ρ may be found from Emden's[2] formulae, or we may use the values given by Eddington[3] for his typical star ($\mathcal{M}=1.5$, $\rho_m=0.002$), taking care to modify those for T, which contain the factor m, to make them conform with the degree of ionization expressed by (43) and its alternate. Since T contains both m

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[1] Professor Paul Epstein of the California Institute of Technology calls my attention to an error in Eggert's expression for K (*op. cit.*, p. 573). A factor 8^6 has been omitted. It has little effect, however, on equation (43), the absolute term being increased from 2.32 to 2.68.

[2] *Gaskugeln*, p. 97, 1907. Eddington has called attention to the misprint in the formula for θ^3. The last factor in the denominator should be squared.

[3] *Astrophysical Journal*, **48**, 213, 1918. Unless I am in error, Eddington's values for the temperature have been computed with $m=2$, whereas the text seems to imply that 2.8 has been used. If so, the tabular values should be increased by 40 per cent.

and β as factors, the total pressure P depends upon neither the value of the radiation-pressure nor the degree of ionization. T, however, depends upon both, and, moreover, β also depends upon m (see equation [42]).

For the removal of 8 and 16 electrons, respectively, the values of m become 6.2 and 3.3. Corresponding to these, for $\mathcal{M}=1.5$, $\beta=0.496$ and 0.766. With these data we find the values of P and T given in the second, third, and fifth columns of Table XX

TABLE XX

IONIZATION AND STELLAR TEMPERATURES, IRON IN EDDINGTON'S TYPICAL STAR

r	Log P (Atm)	−8 Electrons $m=6.2$		−16 Electrons $m=3.3$		Percentage of Mass Outside r
		T	T_i	T	T_i	
0.......	7.40	8.7×10^6	8.0×10^5	7.1×10^6	2.6×10^6	100
1.......	7.13	7.4	6.7	6.1	2.4	88
2.......	6.47	5.0	4.9	4.1	2.0	48
3.......	5.63	3.1	3.4	2.5	1.6	18
4.......	4.69	1.8	2.5	1.5	1.3	5
5.......	3.59	0.96	1.8	0.79	1.1	0.7
6.......	1.98	0.39	1.2	0.32	0.8	0.005

Radius of star $=6.9$, $\mathcal{M}=1.5$, $\rho_m=0.002$.

for various points along the radius. The unit for r is such that the distance from the center to the surface is 6.9. Substituting the values of P into Eggert's formula, we find the ionization temperatures given in the fourth and sixth columns of the table. These temperatures, required for the removal of 8 and 16 electrons, respectively, may be compared with the corresponding stellar temperatures. The latter, which change little with the assumed change in m, are approximately ten times the temperature required to separate 8 electrons from the nucleus, and of the same order as the ionization temperatures corresponding to the removal of 16 electrons. The decrease in pressure toward the surface compensates for the fall in temperature to such an extent that the ionization should be nearly constant throughout the star. It is only within a thin shell near the surface, which contains an insignificant fraction of the mass (see last column, Table XX), that m can rise much above the value 3.3.

Eggert's formula is an extrapolation and involves assumptions which cannot at present be directly controlled; but its general indications are in close agreement with Eddington's conclusions. Even admitting the various underlying assumptions, the result applies only to a star consisting wholly of iron; but one can scarcely doubt that the mean values of m for all the elements actually present must be small and nearly constant for all points within the star except those close to the surface.

It seems worth while also to consider the relation of ionization to stellar temperature in giant stars of constant mass but differing spectral types, in order to gain some idea of the permanence of this relation as a star contracts under the action of gravitation.

Consider stars of Eddington's critical mass $\mathcal{M}=2$, which, as we have seen, are those occurring most frequently among the late-type giants. It will be sufficient to calculate the central temperatures and pressures, and in view of the preceding results it is clear that we may use $m=3.3$. The corresponding value of β is 0.70. Using Emden's formula as before (remembering that for the type of equilibrium considered $\rho_c=54.25\,\rho_m$), we find the results for T_c and P_c given in Table XXI. The values of M and $\log \rho_m$ in the second and third columns were read from the diagram of Figure 3. The ionization temperatures for the removal of 16 electrons are in the last column of the table.

TABLE XXI

CENTRAL TEMPERATURES AND PRESSURES, AND IONIZATION TEMPERATURES FOR IRON
($\mathcal{M}=2$, $m=3.3$)

Sp.	M	Log ρ_m	Log T_c	Log P, (Atm)	T_c	T_i
Ma.............	0.0	−5.1	6.10	4.3	1.3×10^6	1.2×10^6
K5.............	+0.7	−4.1	6.43	5.6	2.7	1.5
Ko.............	+1.1	−3.1	6.77	7.0	5.9	2.1
G5.............	+0.8	−2.7	6.90	7.5	7.9	2.6
Go.............	+0.7	−2.2	7.07	8.2	11.7	3.3
F5.............	+2.0	−1.0	7.47	9.8	29.5	7.0

These data represent the sequence of changes from an early stage of development, on the basis of Russell's theory, through to the dwarf stage, where the density is so large that Boyle's law is no longer accurately obeyed. The increases in T_c, ρ and P_c are approximately 20, 10,000, and 300,000 fold, respectively. In spite of these extraordinary changes, the ionization temperatures remain of the same general order as those at the center of the star throughout its development. Initially, the two temperatures are sensibly equal, and the gain in T_c over T_i is so slight that any further decrease in m must be small. The complete removal of the third sheath of electrons, for example, would give $m=2.2$.

Even for stars of the lowest density, such as Antares or Betelgeuse, conditions are very similar to those illustrated by the first line of Table XXI. Low density means relatively low central temperature and pressure, but this factor is compensated by the large masses which characterize the very luminous giants, so that the relation of ionization to stellar temperatures is much the same as for the objects already examined.

It may be remarked incidentally that the values of T_c in Table XXI throw some light upon the behavior of the effective radiating layers in stars of different types. The increase in T_c is approximately ten times that in the effective temperature. But for the type of equilibrium in question the temperature gradient is independent of the density, that is, the temperature increase for homologous points along the radius should be constant. The explanation must be that the thickness of the radiating layer in an Ma giant, because of the enormous difference in density, is very much greater than that for an F5 dwarf. As development progresses, radiation from the deeper and hotter strata is absorbed as a result of increasing density, and, in consequence, the increase in effective temperature is less rapid than would otherwise be the case.

On the basis of the preceding evidence we must expect a high degree of ionization at the earliest stages of stellar development with which we are familiar, and comparatively little decrease in the value of m as the development proceeds. The Bohr atom affords the only foundation we have for estimates such as have been made, and it is not certain that the results are of the right order of magnitude. One consequence of their acceptance, however, should not be overlooked, and that is the total amount of energy required for ionization. For the removal of 8 electrons this equals 1.8×10^7 calories per gram atom,[1] corresponding to an ionizing potential of about 800 volts. The mass of Eddington's typical star is approximately 2.9×10^{33} grams. The removal of 8 electrons from all the atoms in this mass would therefore require

$$U_0=\frac{2.9\times10^{33}\times1.8\times10^7\times4.2\times10^7}{56}=4\times10^{46}\text{ ergs,}\qquad(45)$$

while for the removal of the second sheath of 8 electrons a quantity of the order of ten times this amount, or

$$U_0=4\times10^{47}\text{ ergs}\qquad(46)$$

would be necessary.

Now the total energy generated by gravitational contraction of the star from a state of infinite diffusion is[2]

$$\Omega=\frac{3(\gamma-1)}{5\gamma-6}\,\frac{G\mathcal{M}^2}{R}\qquad(47)$$

where γ is the ratio of specific heats, G the gravitational constant, 6.66×10^{-8} cm³/grams sec.², and R the radius, 7×10^{11} cm; γ lies between the limits 4/3 and 5/3, whence the fractional coefficient on the right has the limiting values 3/2 and 6/7. At a maximum, therefore, which also corresponds to Eddington's model,

$$\Omega=\tfrac{3}{2}\cdot\tfrac{2}{3}\cdot\frac{10^{-7}\times9\times10^{66}}{7\times10^{11}}=1.3\times10^{48}\text{ ergs.}\qquad(48)$$

Of this total the etherial energy alone,[3] namely,

$$(1-\beta)\Omega=0.234\,\Omega=3.0\times10^{47}\text{ ergs,}\qquad(49)$$

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[1] Eggert, *loc. cit.*

[2] Emden, *Gaskugeln*, p. 125, 1907.

[3] Eddington, *Monthly Notices*, **79**, 23, 1918. Anderson, with whom I have had much stimulating discussion during the preparation of this paper, calls my attention to the fact that ionization cannot be produced by the collision of electrons with atoms without violating the Nernst formula of equilibrium, which is well established on thermodynamical grounds. This equation depends upon the pressure, whereas the hypothesis of ionization by collisions alone leads to a condition for equilibrium which is independent of pressure. If the effective energy is assumed to be partly radiant and partly kinetic, the pressure would enter, but not in the manner required by Nernst's equation. Since the internal kinetic energy of the atoms is not available, the radiant energy alone can be operative (as in the case of the photoelectric effect, for example). See also Milne, *Observatory*, **44**, 269, 1921.

seems to be available for ionization. This is of the same order as U_0, the energy of dissociation given in (46). Admitting, therefore, the validity of Bohr's theory and the applicability of the Nernst formula of equilibrium, the insufficiency of the gravitational theory presents itself in a new form. As an alternative, the question is perhaps raised as to whether the giant stars, even in their very earliest stages of development, are ever in a state other than one of high dissociation.

The calculation of the stellar temperatures in Tables XX and XXI neglects altogether the depletion of the available supply of energy by ionization. Were gravitation the only source, we could expect neither the temperatures nor the degree of ionization indicated by the discussion; but, instead, an equilibrium state which would be established for something less than the extremes of temperature and ionization shown by the tables.

18. COMPARISONS WITH EDDINGTON'S FORMULAE

The assumption concerning m which underlies equations (41) and (42) has been discussed in the preceding section. These formulae also assume that the outflow of energy across any surface of unit area is proportional to the gravitational acceleration at that point ($H \propto g$), and that the mass coefficient of absorption, k, is a constant. The energy assumption can also be stated in the form that the outflow throughout the star per unit of mass is constant. The source of the energy is not specified; it may be derived from gravitational contraction or any other source capable of providing an adequate supply. Assuming further that the transfer of energy is by radiation and not by convection, the conditions for equilibrium are found to be the same as those for adiabatic equilibrium with a ratio of specific heats equal to 4/3. It then follows through (41) and (42) that the bolometric magnitude of a star remains constant so long as it behaves as a perfect gas.[1]

A giant star in its development, however, presumably follows closely one of the equal-mass lines in the upper part of Figure 1, for the selection which enters along the dwarf branch has comparatively little influence on the average mass of the giants which come under observation. Observations indicate, therefore, that the absolute magnitude is not independent of the stage of development. The change is even larger than shown in Figure 3, for, as already stated, the visual absolute magnitudes must be corrected by the quantities given on page 194 to make them comparable with the bolometric magnitudes calculated by (41).

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[1] As Jeans points out, *Monthly Notices*, **79**, 319, 1919, this general result follows at once, by simple considerations, from the assumption $H/g=\text{const}$. The nature of the dependence of radiation on mass and other characteristics is, however, another matter, less easily disposed of.

The question now arises as to the particulars in which the underlying assumptions must be modified in order to effect a closer agreement with observations. Such consideration as can be given here may be prefaced by the remark that Jeans,[1] starting from the assumptions of radiative conduction, gravitational contraction, and constant k and m, finds that the radiation of a star will remain constant as long as Boyle's law is obeyed. This is also the case if the star has sources of energy other than gravitational contraction, provided the rate of generation per unit mass is independent of the time. This is Eddington's result without his assumption that $H \propto g$; but the functional dependence of radiation upon the mass cannot be specified. Jeans also finds that H/g will be constant throughout the mass of gas only when the mass is so large that radiation-pressure predominates and β may be put equal to zero. For the ordinary run of stellar masses, however, H/g will not be constant within the star, but for a given point will be independent of the time, provided the only source of energy is gravitational.

Eddington[2] himself points out that H/g cannot be rigorously constant within the star, for, assuming gravitational contraction through a series of homologous states and a temperature- and pressure-distribution corresponding to adiabatic equilibrium with $\gamma=4/3$, the outflow of energy per unit mass would increase about 70 per cent from the surface to the center.

In view of a probable variation of H/g of the order indicated, the original assumption gives close agreement with observations. The matter can be tested by comparing the calculated values of M corresponding to different values of $\mathcal{M}$ with that shown by Figure 3 for stars of the same spectral type. By proceeding in this manner we avoid disturbances arising from variations in *k* and *m*, since for stars of the same type these quantities should be less subject to change.

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[1] *Loc. cit.*

[2] *Monthly Notices*, 77, 599, 1917.

The results are shown in Table XXII. This table gives the deviations of the observed values of *M* (Fig. 3) from those calculated by Eddington, which are in the second column of the table. Zero point corrections have been applied to the latter to make the mean O−C for each type equal to zero. There is a large progressive change for the F stars, but the agreement improves with advancing type and, with the exception of one difference, is practically perfect for the K and M stars. From this comparison it is clear that the error in the assumption $H/g=$const. is not very serious when we consider stars of a specified type.

TABLE XXII

OBSERVED AND CALCULATED (EDDINGTON) VARIATION OF ABSOLUTE MAGNITUDE WITH MASS

$\mathcal{M}$	CAL. M	O−C				$\mathcal{M}$	CAL. M	O−C			
		Fo	Go	Ko	Ma			Fo	Go	Ko	Ma
1.5	−0.3	.....	+1.5	+0.7	+0.3	6	−2.9	+0.3	−0.3	−0.1	0.0
2	−0.9	.....	+0.9	+0.3	+0.2	7	−3.1	−0.1	−0.5	−0.1	0.0
3	−1.7	.....	+0.5	0.0	0.0	8	−3.3	−0.5	−0.6	−0.2	0.0
4	−2.2	+1.4	+0.1	0.0	0.0	9	−3.5	−0.7	−0.8	−0.3	0.0
5	−2.6	+0.7	−0.1	0.0	0.0	10	−3.7	−1.0	−0.9	−0.3	0.0

The lack of parallelism of the equal-mass lines with the axis $M=0$ is difficult to account for because the assumption of constancy for both *k* and *m* enters here, as well as that concerning the flow of energy. In general the luminosity for any given mass decreases with increasing temperature. For mass 4, as an illustration, the bolometric absolute magnitude increases from −3.1 at Ma to +0.3 at Fo. This is in the direction corresponding to decreasing values of *m*, and one might expect much of the change to be accounted for by increasing ionization. The evidence of section 17, however, is against any considerable variation in *m*. Even in the earliest visible stages of development we should not anticipate a value much in excess of 3.3, and unless there is a large amount of nuclear disintegration, which seems improbable, the value of m cannot fall below 2. It is easily shown that a decrease from $4\sqrt{2}$ to 2 would cause an increase in M of about one magnitude, which is only a third of the amount required. The admission of much larger initial values of m is insufficient to bridge the gap, for (42) shows that $1-\beta$, and hence the luminosity, are not much affected as m is increased.

The behavior of k is obscure. Although the conditions within the star which determine the degree of absorption must be analogous to those governing the absorption of X-rays, it should be borne in mind that the effective value of k is that corresponding to the outermost layers of the star where the temperature is very much lower than in the interior. Even near the surface the opacity is large, but we have no data as to its variations that are certainly applicable.

This leaves the larger part of the increase in the absolute magnitude unaccounted for, and it is not clear as to how the discrepancy is to be explained. If the contraction is homologous and the energy gravitational, or gravitational plus a source independent of the time, we should not expect variations in outflow with advancing development other than those caused by changes in k and in m.

As an explanation, changes in m that would be sufficient seem to be excluded; those of k are in doubt. There remain, however, the assumptions underlying the outflow of energy. The source of energy almost certainly is not wholly gravitational, and if not, with almost equal certainty, is not independent of the time. The outstanding questions therefore lead directly into the realm of atomic physics, and it seems more than ever likely that the problem of the stars will find its solution in the solution of the problem of the atom.

In conclusion, attention may be directed to the relation of mass to luminosity along the dwarf branch. Here Boyle's law is in general no longer obeyed. It will be noted, however, that the density increases slowly until the late types are reached, which suggests that the departure from the conditions of a perfect gas may affect the behavior of stars of different spectral types much alike, so that a formula giving the relation of luminosity to mass for the giants, with the exception of a constant, would also hold approximately for the dwarfs.

The matter is easily tested by calculating $\mathcal{M}$ from (41) for the values of L corresponding to the absolute magnitudes given in the second column of Table IV. Choosing the constant factor so that the mean of the values thus calculated agrees with the observed mean mass, we have the results in Table XXIII. With the exception of a spectral interval at either end of the series, where the deviations from the average density are large, the agreement is good.

TABLE XXIII

VALUES OF MASS ON DWARF BRANCH

Sp.	Obs.	Cal.	Sp.	Obs.	Cal.
Bo	10	18	Go	1.0	1.0
B5	8.3	10.5	G5	0.76	0.76
Ao	6.0	6.6	Ko	0.68	0.62
A5	4.0	4.1	K5	0.62	0.48
Fo	2.5	2.5	Ma	0.59	0.26
F5	1.5	1.5			

I must express my grateful acknowledgments to Miss Joyner and Miss Richmond, of the Computing Division, for their efficient assistance in the extensive tabulations and calculations involved in the preceding discussion.

MOUNT WILSON OBSERVATORY
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ON THE CALCULATION OF MASSES FROM SPECTROSCOPIC PARALLAXES[1]

By HENRY NORRIS RUSSELL[2]

ABSTRACT

Calculation of masses of stars from spectroscopic parallaxes.—It appears probable that the line intensities upon which spectroscopic parallaxes are based are *functions of the temperature and density of the star's atmosphere.* If this were exactly true, all stars of the same surface brightness and density would have the same spectroscopic absolute magnitude, and *the masses, computed from the spectroscopic parallaxes, would come out the same for all the stars* of such a group (whatever the dispersion among their actual masses) and equal to the geometrical mean of the latter. To obtain a reliable measure of the dispersion in mass among binary stars, parallaxes must be determined in some other way.

Spectroscopic and dynamical parallaxes.—It follows that the spectroscopic parallaxes and the dynamical parallaxes (derived on the assumption that the mass of a binary system is equal to the mean value for stars of its spectral type and absolute magnitude) are *systematically equivalent to one another,* and really rest on the same physical relationships and assumptions.

Dispersion of mass among visual binaries (dwarf stars).—By an indirect method, depending on Strömberg's comparisons of spectroscopic and trigonometric parallaxes, Seares's conclusion that the dispersion is small is confirmed. The probable error of dispersion of log $\mathfrak{M}$ appears to be less than ±0.2, but cannot be exactly determined.

The very small values obtained by Seares in the preceding paper[3] for the dispersion in mass among dwarf stars suggest the following explanation.

It appears very probable, from physical considerations, that the spectral type of a star is determined by the temperature T of its outer atmosphere, while the characteristics associated with the absolute magnitude depend also, and mainly, upon the density of the atmosphere, ρ'. If M_s is the absolute magnitude, determined spectroscopically, we may then write

$$M_s = F(T, \rho'). \qquad (1)$$

It is also probable that the surface brightness j depends almost entirely on the temperature, and that the density of the atmosphere above the visible surface is a function of the mean density ρ (though not necessarily proportional to it). We have then

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[3] *Mt. Wilson Contr.,* No. 226.

$$M_s = f(j, \rho). \tag{2}$$

It may be that M_s depends to some degree on other variables, but it is probable that their influence is slight. The form of the function f cannot be predicted by theory, but can be found empirically when sufficient data regarding stellar masses and densities are available.

Assuming the relation for the moment to be exact, and introducing it into equation (22) of Seares's paper, we have

$$\log \mathcal{M}_s = \log \rho + 0.6j - 0.6f(j, \rho) + 2.77. \tag{3}$$

Hence $\mathcal{M}_s$ the mass corresponding to the spectroscopic parallax is itself a function of ρ and j alone. Like (22) this is a general result, holding good no matter how the mass is calculated, so long as the formula involves the absolute magnitude and our assumptions about the latter are correct.

Stated in words, this signifies: *If the spectroscopic absolute magnitude of a star depends only on its surface brightness and density, the masses computed from the spectroscopic parallaxes will be identical for all stars which have the same density and surface brightness, no matter how different their actual masses may be.*

In still other words, the spectroscopic parallax (on our assumptions) will be correct only in the case of stars having a certain average value of the mass. For larger or smaller masses there will be a systematic error in the absolute magnitudes, making the former too faint and the latter too bright, which will alter the spectroscopic parallax to just such a degree as to conceal entirely the real differences in mass.

If the spectroscopic calibration curves are adjusted so as to give a correct value for the mean absolute magnitude of the stars, the "spectroscopic mass" will be the geometrical mean of the individual masses, since, if j and ρ are constant, M varies proportionally to $\log \mathcal{M}$.

To determine the dispersion in mass among visual binaries, it is therefore necessary to have recourse in some way to parallaxes other than spectroscopic. One way of doing this is through Strömberg's[1] discussion of the errors of the spectroscopic parallaxes—which he has determined by comparison of individual spectroscopic and trigonometric parallaxes, allowing for the errors of the latter. Let the probable error thus found for log π be $\pm r$. Three sources contributory to this error may be distinguished: (1) the error just discussed, arising from differences between the masses of individual stars of the same surface brightness and density; (2) the error arising from other physical causes, on account of which the functional relation (2) is not exactly true; and (3) errors of observation. If we call the amounts of these, expressed as probable errors of log π, respectively, r_m, r_p, and r_e, we will have

$$r^2 = r_m^2 + r_p^2 + r_e^2 .$$

If, on the other hand, we compute individual masses as Seares has done, the error r_m will be without influence on these masses, while r_p and r_e will affect log $\mathcal{M}$ by 0.6 of their amounts, and there will be an additional source of error, due to the imperfections of the orbits and also to errors in the assumed ratio of the masses of the components, which we may call $\pm r_c$. If then r' is the probable error of distribution, which Seares calls $r(\Delta \log \mu)$, we have

$$\begin{aligned} r_m^2 + r_p^2 + r_e^2 = r^2 &= (0.36)^2 \\ 0.36(r_p^2 + r_e^2) + r_c^2 = r'^2 &= (0.22)^2 \\ r_m^2 &= 2.78\, r_c^2 - 0.004 . \end{aligned}$$

This would indicate that the actual dispersion in log $\mathcal{M}$ (which corresponds to the probable error $\pm 0.6\, r_m$) is very small—being substantially equal to the spurious dispersion which arises from errors in the orbital elements of the binaries.

The observed values of r and r' are, however, subject to some uncertainty, and it is possible that some other concealed correlation may still be making the agreement appear better than it should.

It is obvious, however, that a better way of determining the dispersion in mass among visual binaries of the same spectral type will be a direct comparison of "hypothetical" or dynamical parallaxes, computed with a mean value of the mass, with trigonometric parallaxes. In this case there seems to be no reason to fear a correlation of the sort discussed above.

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[1] *Mt. Wilson Contr.*, No. 199, p. 15; *Astrophysical Journal*, 53, 13, 1921.

It may be noticed that the dynamical parallax of a binary, calculated on the assumption that the principal star has the mass $\mathcal{M}_s$ derived from spectroscopic parallaxes of stars of similar kind, and the spectroscopic parallax itself, must agree exactly in every case (barring the effects of errors of observation and the small uncertainty arising in the estimate of the ratio of the mass of the system to that of the brighter component), so that it may fairly be said that spectroscopic and dynamical parallaxes are systematically equivalent to one another, so long as the mass used in computing the dynamical parallaxes is taken as a function of both absolute magnitude and spectral type. The two then rest essentially on the same physical assumptions—though this is far from obvious at first sight. If this conclusion is correct, spectroscopic parallaxes are unfitted by their very nature to give us information about the differences in mass between systems which have similar spectra (with regard to the "absolute magnitude lines" as well as other characteristics).

When a direct determination of the dispersion in mass among visual binaries has been made—for example, from the trigonometric parallaxes—it will be possible to obtain a valuable control upon the theory which atttributes the spectroscopic phenomena in question to differences in density.

Mount Wilson Observatory
February 16, 1922

ON THE UPPER LIMIT OF DISTANCE TO WHICH THE ARRANGEMENT OF STARS IN SPACE CAN AT PRESENT BE DETERMINED WITH SOME CONFIDENCE[1]

By J. C. KAPTEYN[2] and P. J. VAN RHIJN

ABSTRACT

If both $\phi(M)$ and N_m, the luminosity-curve and the total numbers of stars for each apparent magnitude, were completely known, the star-density could be found for any distance from the sun. The paper deals with the question: What can be reached now that our knowledge of these quantities is limited?

The conclusion is that the incompleteness of our knowledge of $\phi(M)$ is of little importance, but that the contrary is true for N_m. Even if we assume the law of the densities, formula (2), to hold for all values of m, the constants cannot be determined with the accuracy needed for the derivation of the densities at considerable distances. The limits of distance for which the density can be determined within 25 or 40 per cent of its amount were finally found on the supposition that the values of N_m are known to $m=14$ and $m=20$, respectively, and are given in Table XII. With the limit $m=17$, visual, for which we may expect complete values of N_m in the near future, the densities should become pretty reliable for the whole of the domain within which the density exceeds 0.1 of that near the sun.

The conclusions of the present paper must be considered only provisional, mainly for the two following reasons: (*a*) The extinction of light in space has been assumed to be inappreciable; (*b*) No use has been made of a recent substantial improvement in the average parallaxes of the more distant stars.

A more definitive solution now being made at the Laboratory of Groningen will duly take into account these two points. The progress of such a solution being necessarily very slow, the present provisional treatment was undertaken mainly for the purpose of finding the most promising lines for conducting the further treatment of the sidereal problem.

The points which will be successively discussed are as follows:

I. If for any determined part of the sky we know *completely* (*a*) the luminosity-curve $y=\phi(M)$ and (*b*) $N_m=$ the total number of stars of magnitude $m-1/2$ to $m+1/2$ *per ten thousand square degrees*, then we can find the star-density $\Delta(\rho)$ at any distance ρ.

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The solution is contained in *Mount Wilson Contribution*, No. 188, formula (13). The difficulty of the problem lies exclusively in the required completeness.

II. Half of the difficulty may be safely ignored, for it can be shown that at present and for a long, long while to come we may certainly adopt the form

$$\phi(M)=\frac{H}{\sqrt{\pi}}e^{-H^2(M-M_0)^2}=e^{p+qM+rM^2} \tag{1}$$

as representing the luminosity-curve completely, that is, for all values of M from $-\infty$ to $+\infty$.

We will here adopt the values of the constants p, q, r, given in *Contribution* No. 188, modified in accordance with the changes in definitions to be explained presently.

III. There thus remains only the difficulty due to our incomplete knowledge of N_m. We will show what limitation is thereby introduced in the solution mentioned in I, and we will further try to show how the limits can be extended by future countings of the fainter magnitudes.

In this inquiry, proceeding from the simple to the more difficult, we will consider three cases:

a) We will *assume* that the formula

$$N_m=\frac{h}{\sqrt{\pi}}e^{-h^2(m-m_0)^2}=2e^{a+bm+cm^2} \tag{2}$$

holds not only for the magnitudes for which we have the necessary data but also for all fainter magnitudes.

b) We will drop the latter supposition, that is, in accordance with the results found thus far, we will adopt the error-curve (2) for the actually observed N_m, but will assume nothing about the fainter magnitudes which have as yet not been investigated.

c) We will consider the most general case, namely, that the actually observed values of N_m themselves are *not* distributed in an error-curve.

IV. Having thus found the limits within which a fairly reliable determination of the arrangement of stars in space can be made, we will introduce what we think must be considered a plausible supposition, by which these limits, at least for the lower galactic latitudes, can be somewhat farther extended.

Before entering on a discussion of these points it will be necessary to premise the following remarks:

a) Cluster observations by Shapley have fairly well proved the absence of any selective extinction of light in space, at least for distances like those considered in the present discussion. It does not follow that there is no appreciable general extinction of light, and the matter will be carefully considered in our definitive solution. Still, in view of Shapley's observations, we feel less hesitation in making the assumption of no appreciable extinction the basis of this provisional solution.

b) Up to the present, the luminosity-curve has expressed the number of stars of each magnitude contained in a specified volume of space in the neighborhood of the sun. It would certainly have been much more natural and convenient had it been given the form of a true frequency-curve, that is, one showing the fraction of all existing stars belonging to each absolute magnitude. This, of course, is not rigorously possible as long as our observations do not embrace all the stars, even the very faintest in existence. But since the luminosity-curve has been derived (in *Contribution* No. 188) over a range of more than twenty magnitudes and since its coincidence with a normal error-curve throughout this entire interval, which extends far beyond the maximum, is truly astonishing, it would seem very desirable to introduce the *assumption* that the best-fitting error-curve represents the luminosity-curve over its whole extent. In this way it becomes possible to define the luminosity function as a true frequency-curve.

Henceforth we will assume, therefore, that the luminosity-curve, over its whole extent, is perfectly representable by equation (1), which differs from formula (11) in *Contribution* No. 188, only in being divided by the factor $A=0.0451$, the number of stars per cubic parsec in the neighborhood of the sun. If later on it should appear that this assumption deviates sensibly from the truth, the final results will still be unaffected. The ordinates of the curve will have only to be multiplied by a certain factor, while the densities will be divided by that same factor.

c) The parsec is again taken as a unit of distance, and in accordance with this definition the absolute magnitude of a star will equal its apparent magnitude as seen from a distance of one parsec.

TABLE I*a*

LUMINOSITY-CURVE

(Equation [5])

M	$\phi(M)$	$\log \phi(M)$		
+10	0.00229	7.359		
			470	
+ 9	.00675	7.829		70
			400	
+ 8	.0169	8.229		68
			332	
+ 7	.0365	8.561		69
			263	
+ 6	.0667	8.824		69
			194	
+ 5	.104	9.018		70
			124	
+ 4	.139	9.142		68
			056	
+ 3	.158	9.198		69
			013	
+ 2	.153	9.185		69
			082	
+ 1	.127	9.103		70
			152	
0	.0893	8.951		68
			220	
− 1	.0538	8.731		70
			290	
− 2	.0276	8.441		67
			357	
− 3	.0121	8.084		70
			427	
− 4	.00456	7.657		70
			497	
− 5	.00145	7.160		68
			565	
− 6	.000394	6.595		69
			634	
− 7	.0000914	5.961		69
			703	
− 8	.0000181	5.258		70
			773	
− 9	.00000305	4.485		69
			842	
−10	0.00000044	3.643		

$M = m + 5 \log \pi$

d) Finally, the star-density $\Delta(\rho)$ now means the total number of stars from the brightest to the faintest in one cubic parsec.

Under these stipulations we have in (1), in accordance with *Contribution* No. 188,

$$H=0.2818, \qquad M_0=2.693 \tag{3}$$

$$p=-2.413^{1} \qquad q=+0.4278 \qquad r=-0.07944 \tag{4}$$

$$\log\phi(M)=-1.049+0.1858\,M-0.03450\,M^2 \tag{5}$$

$$M=m+5\log\pi=m-5\log\rho. \tag{6}$$

The numerical values of (5), that is, of the luminosity-curve according to the present definitions, have been tabulated in Table I*a*.

For convenience we give in Table III the values of the constants of *Contribution* No. 188, p. 13, changed to agree with our

TABLE I*b*

log Δ (ρ) (FORMULAE OF TABLE IV)

log ρ	Galactic Latitude				
	0°	30°	60°	90°	40°–90°
1.0	8.65−10	8.65−10	8.65−10	8.65−10	8.65−10
1.2	.65	.65	.65	.65	.65
1.4	.65	.65	.65	.65	.64
1.6	.65	.65	.65	.65	.61
1.8	.65	.65	.65	.65	.57
2.0	.65	.54	.50	.52	.49
2.2	.65	.46	.42	.45	.40
2.4	.65	.33	.25	.26	.25
2.6	.55	.16	.00	7.95	.03
2.8	.41	7.93	7.67	.51	7.73
3.0	.21	.64	.25	6.95	.32
3.2	7.97	.31	6.75	.27	6.85
3.4	.69	6.92	.15	5.46	.31
3.6	.35	.48	5.48	4.53	5.70
3.8	6.97	5.98	4.72	3.48	.01
4.0	.54	.44	3.88	2.30	4.24
4.2	.07	4.84	2.95	1.00	3.40
4.4	5.54	.19	1.93	....	2.48
4.6	4.97	3.49	0.84	....	1.49
4.8	2.35	2.73	....	....	0.42
5.0	3.68	1.92	....	....	....

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[1] Differing from the corresponding value of p in *Contribution* No. 188, p. 15, by the amount $\frac{1}{\text{Mod.}}\log(0.0451)=3.099$.

TABLE II

DISTRIBUTION IN DISTANCE. FRACTIONS OF TOTAL NUMBER OF STARS (W, Equation [12])

log ρ	$m=8$	9	10	11	12	13	14	15	16	17	18	19	20
	Galactic Latitude = 0°												
−∞ to 2.8	0.7779	0.6761	0.5596	0.4382	0.3224	0.2213	0.1409	0.0835	0.0458	0.0231	0.0107	0.0046	0.0018
2.8 to 3.0	.1223	.1590	.1886	.2037	.2002	.1804	.1478	.1107	.0753	.0470	.0268	.0138	.0065
3.0 to 3.2	.0640	.0972	.1344	.1689	.1950	.2048	.1961	.1708	.1366	.0994	.0658	.0400	.0220
3.2 to 3.4	.0256	.0456	.0734	.1085	.1456	.1781	.1994	.2040	.1900	.1610	.1252	.0886	.0570
3.4 to 3.6	.0080	.0164	.0309	.0531	.0833	.1197	.1568	.1864	.2026	.2016	.1825	.1505	.1131
3.6 to 3.8	.0018	.0046	.0100	.0202	.0370	.0617	.0942	.1316	.1670	.1931	.2041	.1975	.1737
3.8 to 4.0	.0003	.0009	.0026	.0058	.0125	.0244	.0439	.0711	.1053	.1428	.1764	.1982	.2038
4.0 to 4.2	.0001	.0002	.0004	.0013	.0032	.0075	.0156	.0296	.0513	.0811	.1166	.1538	.1850
4.2 to 4.4	.0000	.0000	.0001	.0003	.0007	.0017	.0042	.0095	.0192	.0353	.0596	.0916	.1286
4.4 to 4.6	......	......	.0000	.0000	.0001	.0004	.0009	.0023	.0055	.0118	.0233	.0417	.0686
4.6 to 4.8	......	......	......	......	.0000	.0000	.0002	.0004	.0012	.0031	.0070	.0147	.0283
4.8 to 5.0	......	......	......	......	......	......	.0000	.0001	.0002	.0006	.0017	.0040	.0090
5.0 to 5.2	......	......	......	......	......	......	......	.0000	.0000	.0001	.0003	.0009	.0021
5.2 to 5.4	......	......	......	......	......	......	......	......	......	.0000	.0000	.0001	.0004
5.4 to 5.6	......	......	......	......	......	......	......	......	......	......	......	.0000	.0001
	Galactic Latitude 40°–90°												
−∞ to 2.8	0.9190	0.8696	0.8015	0.7161	0.6168	0.5079	0.4001	0.2979	0.2097	0.1397	0.0873	0.0513	0.0281
2.8 to 3.0	.0569	.0860	.1213	.1585	.1920	.2168	.2263	.2207	.1991	.1671	.1298	.0944	.0633
3.0 to 3.2	.0188	.0330	.0545	.0830	.1178	.1551	.1894	.2145	.2267	.2220	.2017	.1706	.1337
3.2 to 3.4	.0044	.0093	.0177	.0317	.0520	.0802	.1143	.1517	.1863	.2131	.2262	.2226	.2042
3.4 to 3.6	.0008	.0018	.0042	.0087	.0168	.0300	.0499	.0774	.1113	.1480	.1837	.2111	.2252
3.6 to 3.8	.0001	.0003	.0007	.0017	.0039	.0081	.0157	.0285	.0480	.0743	.1078	.1446	.1806
3.8 to 4.0	.0000	.0000	.0001	.0003	.0006	.0016	.0036	.0076	.0149	.0271	.0457	.0717	.1045
4.0 to 4.2	......	......	.0000	.0000	.0001	.0003	.0006	.0015	.0034	.0071	.0141	.0256	.0437
4.2 to 4.4	......	......	......	......	.0000	.0000	.0001	.0002	.0005	.0014	.0031	.0067	.0133
4.4 to 4.6	......	......	......	......	......	......	.0000	.0000	.0001	.0002	.0005	.0012	.0029
4.6 to 4.8	......	......	......	......	......	......	......	......	.0000	.0000	.0001	.0002	.0004
4.8 to 5.0	......	......	......	......	......	......	......	......	......	......	.0000	.0000	.0001

modified definitions, that is, with (4) instead of with the values previously given for p, q, r.

The values in Table III lead to the formulae in Table IV. These have been tabulated in Table I*b*.

With these details premised we will now discuss the several points separately.

TABLE III

Parameter	Galactic Latitude				
	0°	30°	60°	90°	40°–90°
h	−8.929	−8.292	−12.320	−17.418	−10.984
k	+5.705	+5.481	+ 9.320	+14.092	+ 8.120
l	−1.366	−1.508	− 2.441	− 3.542	− 2.171

TABLE IV

Gal. Lat.	Density and Distance		
0°	$\log \Delta(\rho) = -3.878 + 2.478$	$\log \rho - 0.593$	$(\log \rho)^2$
30°	$= -3.602 + 2.381$	-0.655	
60°	$= -5.351 + 4.048$	-1.060	
90°	$= -7.565 + 6.120$	-1.538	
40°–90°	$= -4.770 + 3.526$	-0.943	

POINT I

As already mentioned, the solution of the problem involved in this case is contained in formula (13) of *Contribution* No. 188.[1] Modified in accordance with what precedes, this formula becomes:

$$N_m = 3.046 \int_0^\infty \rho^2 \Delta(\rho) \phi(m - 5 \log \rho) d\rho. \tag{7}$$

From this equation it is at once clear that given N_m and $\phi(M)$ for *all* values of m and M, we can find $\Delta(\rho)$, and this constitutes in reality the solution of our problem. The derivation involves the solution of the integral equation (7). For the case in which $\phi(M)$ has really the form (1) and N_m the form (2), Schwarzschild has given an elegant solution, which, quoted already in *Contribution* No. 188, but modified in accordance with our present definitions, is contained in the following formulae:

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[1] For this formula, in a slightly different form, see *Astronomical Journal*, 24, 27 (No. 566), 1904.

$$\Delta(\rho)=e^{h+k\log\rho+l(\log\rho)^2} \tag{8}$$

$$\left.\begin{aligned} l&=\frac{25cr}{r-c}\\ G^2&=-l-25r\\ R&=\frac{G^2(b-q)}{5r}\\ k&=5q-R-\frac{3}{\mu}\\ h&=a-p+\frac{\log G}{\mu}-\frac{R^2}{4G^2}-2.5203\\ \frac{3}{\mu}&=\frac{3}{\text{Mod}}=6.9078. \end{aligned}\right\} \tag{9}$$

Admitting that (8), which for brevity we will call the Schwarzschild formula, holds for all distances from $\rho=0$ to $\rho=\infty$, it is, of course, easy to determine all such quantities as those which follow (the area is always assumed to be 10,000 square degrees):

N_m = total number of stars of appt. mag. m
$(N_m)_0^\rho$ = the same within the distance ρ
N = total number of stars of all magnitudes together
N_ρ = the same within the distance ρ
M = median = distance within which lie one-half of all the stars
$W=(N_m)_0^\rho/N_m$
$Z=N_\rho/N$.

We find the formulae:

$$N_m=\frac{0.9696\pi\sqrt{\pi}}{\mu G}e^{G^2(\beta^2+\gamma)}=\frac{12.432}{G}e^{G^2(\beta^2+\gamma)} \tag{10}$$

$$(N_m)_0^\rho=WN_m \tag{11}$$

$$W=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{G(\log\rho-\beta)}e^{-z^2}dz \tag{12}$$

$$2\beta G^2=-R-10rm \tag{13}$$

$$\gamma G^2=h+p+qm+rm^2 \tag{14}$$

$$N=\frac{12.433}{\sqrt{-l}}e^{h-\frac{\left(k+\frac{3}{\mu}\right)^2}{4l}}\qquad\left(\frac{3}{\mu}=6.9078\right).\tag{15}$$

As a check on the computations of *h*, *k*, *l*, *N* we have, expressed in terms of *a*, *b*, *c*,

$$N=\frac{\sqrt{\pi}}{\sqrt{-c}}e^{a-\frac{b^2}{4c}}\tag{16}$$

$$\log M=-\frac{k+\frac{3}{\mu}}{2l}\qquad\left(\frac{3}{\mu}=6.9078\right)\tag{17}$$

$$N_\rho=ZN\tag{18}$$

$$Z=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{-l}(\log\rho-\log M)}e^{-z^2}dz.\tag{19}$$

Introducing the constants of Table III into (12), (13), and (14), we obtain

$$\left.\begin{array}{ll}\text{Gal. lat.}=0^\circ & \text{Gal. lat.}=40^\circ\text{ to }90^\circ\\ \quad G=1.833 & \quad G=2.039\\ \quad \beta=1.560+0.1182m & \quad \beta=1.550+0.0956m\end{array}\right\}\tag{20}$$

with the aid of which equation (12) gives at once an insight into the distribution in distance of the stars of each magnitude *m*. The results have been summarized for galactic latitudes 0° and for 40° to 90° in Table II. For the medians we find

$$\left.\begin{array}{ll}\text{Gal. lat. }0^\circ & M=39{,}000\text{ parsecs}\\ 40^\circ\text{ to }90^\circ & 2{,}900\text{ parsecs}\end{array}\right\}\tag{21}$$

Since the Schwarzschild formula does not hold for the smaller distances, results for these have been omitted from the table.

But the supposition that we know $\phi(M)$ and N_m completely, that is, for all values of *M* and *m*, is far from being justified. In the present state of science this incompleteness of our knowledge is very serious in the case of N_m, but of little account in the case of $\phi(M)$. The latter part of this statement constitutes the second point.

POINT II

The practical sufficiency of our knowledge of the luminosity-curve is due to the fact, already alluded to, that it has now been found for so large a range of magnitudes, viz., from the very brightest down to about absolute magnitude $M=+9$.

According to (6) a star of apparent magnitude m will be of absolute magnitude $+9$, when its distance is such that

$$\log\rho=\frac{m-9}{5}.$$

If, therefore, we write the equation (7) in the form

$$\frac{N_m}{3.046}=\int_{\log\rho=-\infty}^{\log\rho=\frac{m-9}{5}}\rho^2\Delta(\rho)\phi(m-5\log\rho)d\rho+\int_{\log\rho=\frac{m-9}{5}}^{\log\rho=+\infty}\rho^2\Delta(\rho)\phi(m-5\log\rho)d\rho \quad (22)$$

the difficulty introduced by incomplete knowledge of the luminosity-curve lies exclusively in the first integral, for the second includes only stars brighter than $M=+9$, for which $\phi(M)$ is known. The difficulty cannot, therefore, be very serious because the first integral involves only stars relatively near the sun. In fact, as will be explained below, the values of $\Delta(\rho)$ within the limits of the first integral may be considered as completely known and its divergence from (8), though considerable, still unimportant for the present question, by a method wholly independent of (7) for all apparent magnitudes to $m=23$ (corresponding to the limit 2.8 for $\log\rho$). It will undoubtedly be a long time before we can expect useful data for N_m beyond $m=23$.

This being so, it is easy to show that there is an overwhelming probability that the first integral in (22) is entirely negligible, at least to the limit $m=23$. We first determine the value of this integral, denoted by V_m, on the supposition that for $M>+9$, $\phi(M)$ is still represented by (1). It is evident that V_m represents the number of stars of apparent magnitude m, absolutely fainter than $M=+9$.

We easily find

$$\frac{V_m}{N_m}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{P} e^{-z^2}dz \tag{23}$$

$$P=\frac{9r-\frac{1}{2}(b-q)-cm}{\sqrt{c-r}}. \tag{24}$$

The constants p, q, r are given by (4); a, b, c in *Contribution* No. 188, p. 13.

Table V, computed by the aid of (23), shows that, if the luminosity-curve (1) were correct over its *whole* length, the stars fainter than absolute magnitude +9 would not contribute to the total number N_{23} in the Milky Way more than 1 star in 16,000, or to those in galactic latitudes 40° to 90° more than 1 star in 300. For N_{22}, N_{21}, etc., the contribution is much smaller still.

TABLE V

VALUES OF V_m/N_m

m	Gal. Lat. 0°	Gal. Lat. 40°-90°
18	0.00000050	0.0000113
19	.00000143	.0000414
20	.00000392	.000139
21	.0000104	.00043
22	.0000262	.00121
23	.0000634	.0032
24	.000148	.0076
27	.00143	.0635
30	0.0095	.267

Therefore, even if beyond $M=+9$, the error-curve (1) should begin to fail very seriously to represent the true frequencies of the absolute magnitudes, it will still lead to values of N_m which will be practically correct to magnitude 23 and probably to even much fainter stars.

Suppose for instance that the frequencies, which up to $M=+9$ are represented with such astonishing closeness by the error-curve (1), should, beyond this magnitude, suddenly become tenfold the frequencies furnished by (1); even then the representation would be quite satisfactory for all values of N_m to $m=23$. Of course the overwhelming probability is that matters are really much more favorable. For we must surely expect that even if the true curve begins to diverge from (1), it will not suddenly show tenfold values. We should rather expect a deviation which for $M = +10$ is still small and only gradually increases, perhaps to very considerable amounts, as we approach fainter and fainter stars. If so, the divergence in the representation of N_m will undoubtedly be still much smaller.[1]

The conclusion from what precedes can only be: as long as observations do not extend beyond $m = 23$, formula (7), in which $\phi(M)$ is given by (1), must be considered as valid. Should they ever reach stars fainter than $m = 23$, a direct extension of the luminosity-curve beyond $M = +9$ may seem desirable. It is easy to see how this could be accomplished. We should require, however, more extensive data on the very faint stars having very considerable proper motions.

POINT III

It now remains to be seen what result for $\Delta(\rho)$ can be deduced from formula (7) with the aid of our *incomplete* data for the values of N_m. First consider the case:

a) We *assume* the form (2) to hold for all magnitudes from the very brightest down to $m = \infty$, referring for less hypothetical considerations to III*b* and III*c*. The problem evidently is: given, $\phi(M)$ and N_m by the formulae (1) and (2) for all values of M and m; required, $\Delta(\rho)$ for all values of ρ.

The solution has been given above in equations (8) and (9). Although theoretically this solution leaves nothing to be desired, we find in most cases that, for moderately large values of ρ, it becomes practically illusory. The reason for this of course is that existing data do not furnish the values of the constants h, k, l, with sufficient precision.

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[1] Perhaps this will not be granted, for of course it is not impossible, *mathematically*, that after $M = +9$, the frequencies instead of continuing to decrease, may begin to increase at such a tremendous rate that the preceding conclusion would no longer hold. Even this possibility must probably be denied. If after $M = +9$ the frequencies should increase at such a rate, the number of very faintly luminous stars near the sun, that is, apparently faint stars with very large proper motion would, I think, exceed what is reconcilable with the observations.

An example, which has a considerable interest of its own, will help to show the difficulty in its true light. As such, we will try to derive the arrangement in space of the stars in the star-cloud between β and γ Cygni. We choose this example because we happen to possess data for part of this region down to photographic magnitude 18.5 (=visual magnitude 17.33). The part in question is that covered by Selected Area (syst. plan) No. 64. The low limit of magnitude is reached by using unpublished results of the Mount Wilson photometric survey together with those in the Harvard Durchmusterung.[1] Simple countings gave the numbers in Tables VI and VII.

TABLE VI

HARVARD DURCHMUSTERUNG

Harv. Photographic Mag.		N'_m for 40'×40'	Comp. by (25)	O−C
Limits	Mean			
8.00 to 8.99..........	8.5	1	2	− 1
9.00 to 9.99..........	9.5	7	7	0
10.00 to 10.99..........	10.5	21	19	+ 2
11.00 to 11.99..........	11.5	42	49	− 7
12.00 to 12.99..........	12.5	143	119	+24
13.00 to 13.99..........	13.5	227	269	−42
14.00 to +14.99..........	14.5	591	570	+21
Total..............	....	1032	....	....

TABLE VII

MOUNT WILSON PHOTOMETRIC SURVEY

M.W. Photographic Mag.		N'_m for 15'×15'	N'_m Reduced to 40'×40'
Limits	Mean		
12.5 to 13.5.................	13.0	20½	146
13.5 to 14.5.................	14.0	43½	309
14.5 to 15.5.................	15.0	97½	693
15.5 to 16.5.................	16.0	200½	1426
16.5 to 17.5.................	17.0	363½	2584
17.5 to 18.5.................	18.0	677½	4817
Total	....	1403	....

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[1] *Harvard Annals* 101.

These tables do not contain even approximately all the stars in the two catalogues. The faintest stars measured on the Harvard plates are 15.75 (phot.), those on the Mount Wilson plates 18.86 (phot.). Stars fainter than 14.99 and 18.50, respectively, were excluded because we cannot be sure that the catalogues are absolutely complete below these limits.

We have first of all to reduce the magnitudes to the same scale and to the same area. The latter reduction has already been made in the last column of Table VII. To find the scale difference the observed numbers N'_m in Table VI were represented by an equation of the form of (2). The result may be written

$$\log N'_m = 2.074 + 0.370(m-12.5) - 0.0145(m-12.5)^2. \tag{25}$$

The values computed by means of this formula are in the fourth column of Table VI; the fifth shows the residuals O−C, whose changes of sign are all that can be desired. With the aid of (25) the Harvard magnitudes were calculated for which the numbers N'_m become equal to the first three values of N'_m in Table VII, viz., 146, 309, 693. The results are 12.75, 13.68, and 14.78, whence we conclude:

	Mt. W.−Harv.
Harv. mag. 12.75=Mt. W. mag. 13.0	$+0^m.25$
Harv. mag. 13.68=Mt. W. mag. 14.0	$+0.32$
Harv. mag. 14.78=Mt. W. mag. 15.0	$+0.22$

There is no indication in the differences Mt.W.−Harv. of change with the magnitude, and we adopt

$$\text{mag. Mt. W.} - \text{mag. Harv.} = \text{Const.} = +0^m.26. \tag{26}$$

By means of (26) the values in Table VII can now be reduced to the scale of Table VI.

A further reduction from the photographic to the visual scale is necessary because the luminosity-curve defined by (1) and (4) gives the frequencies for the visual magnitudes. For this we have used the corrections given in *Groningen Publications*, No. 27, p. 42. The first five columns of Table VIII show the complete reduction.

TABLE VIII

N'_m FOR 40′×40′

PHOTOGRAPHIC SCALE		VIS. SCALE HARV.	AMPL.	m H.S.	N'_m AMPL. 1. MAG.	COMPUTED		O−C	
Harv.	Mt. W.					(27)	(28)	(27)	(28)
8.00 to 8.99	....	7.64 to 8.61	0.97	8.125	1	2	3	− 1	− 2
9.00 to 9.99	....	8.62 to 9.57	.95	9.095	7	7	8	0	− 1
10.00 to 10.99	....	9.58 to 10.52	.94	10.05	22	22	22	0	0
11.00 to 11.99	....	10.53 to 11.47	.94	11.00	44	55	54	− 11	−10
12.00 to 12.99	....	11.48 to 12.41	.93	11.945	154	134	127	+ 20	+27
13.00 to 13.99	....	12.42 to 13.34	.92	12.88	247	296	281	− 49	−34
14.00 to 14.99	....	13.35 to 14.28	.93	13.815	635	629	598	+ 6	+37
....	12.5 to 13.5	11.71 to 12.67	.93	12.175	157	163	155	− 6	+ 2
....	13.5 to 14.5	12.64 to 13.58	.94	13.11	329	358	340	− 29	−11
....	14.5 to 15.5	13.58 to 14.51	.93	14.045	745	748	716	− 3	+29
....	15.5 to 16.5	14.51 to 15.45	.94	14.98	1517	1475	1439	+ 42	+78
....	16.5 to 17.5	15.45 to 16.39	.94	15.92	2749	(2751)	2793	(− 2)	−44
....	17.5 to 18.5	16.39 to 17.33	0.94	16.86	5124	(4835)	5176	(+289)	−52

The fifth and sixth columns give the observational data fully reduced for the present discussion. The numbers in the cloud were first compared with the corresponding numbers for the average Milky Way. It turned out at once that these numbers are all but perfectly proportional, the cloud having about 2.39 times the number for the average Milky Way. This is shown by the seventh column of the table which gives the carefully interpolated values for the Milky Way according to *Groningen Publication*, No. 27, reduced to an area of 40′×40′ and multiplied by 2.39. The agreement with the observed cloud numbers in the sixth column is as good as can be expected. The numbers in the Milky Way in *Groningen Publication*, No. 27, do not extend beyond magnitude 16. For these very faint magnitudes the table gives in parentheses the values furnished by

$$\log N'_m = -4.516 + 0.7242m - 0.0141m^2 \qquad (27)$$

which best fits the numbers of *Groningen Publication*, No. 27, reduced to 40′×40′ and multiplied by 2.39.

On the assumption made in this paragraph, that the formula holds for *all* magnitudes, we conclude that at all distances the star-densities in the cloud are 2.39 times the corresponding densities in the average Milky Way.

Meanwhile we can represent the observed numbers quite as well by a formula somewhat different from (27); for instance, by

$$\log N'_m = -3.951 + 0.6334m - 0.0106m^2. \qquad (28)$$

The representation by this formula is also shown in Table VIII.

By means of the constants in (27) and (28) and those of the luminosity-curve (5), we can at once pass from the distribution of the values N_m to that of the densities by means of formulae (8) and (9). We thus obtain

	Solution (27)	Solution (28)	
h	−8.060	−3.488	
k	+5.719	+2.678	(29)
l	−1.375	−0.882	

The two solutions represent rather widely different arrangements in space. The best way of showing the difference in compact form will perhaps be to compute for both, by formulae (16) and (17),

N = total number of stars for a field of $40' \times 40'$, and

M = median distance below which lie just half of all the stars.

Introducing the values (29), we find,

$$\begin{array}{lll} & \text{Solution (27)} & \text{Solution (28)} \\ M\ldots\ldots\ldots\ldots & 39{,}000 \text{ parsecs} & 272{,}000 \text{ parsecs} \\ N_0^\infty\ldots\ldots\ldots & 579{,}000 \text{ stars} & 3{,}700{,}000 \text{ stars} \end{array}\Bigg\} \tag{30}$$

Identical data thus lead to two solutions for the arrangement in space of the stars in the cloud. In the second, judging by the median values, the star-distances are 7 times those of the first solution. The total numbers[1] are as 1 to 6.5.

Evidently, therefore, the data at present available do not lead to a reliable determination of the real structure of the cloud. The reason clearly lies in the fact that the data of Table VIII are insufficient for a good determination of the constants for N_m in equation (2), and it is readily understood that it is mainly the uncertainty in the position of the maximum to which the failure of the solution is due.

In the notation of (2), this maximum lies at the magnitude

$$m_0 = -\frac{b}{2c} \tag{31}$$

whence for

$$\left.\begin{array}{ll} \text{Solution (27)} & m_0 = 25.68 \\ \text{Solution (28)} & m_0 = 29.88 \end{array}\right\} \tag{32}$$

a difference of over 4 magnitudes!

We are thus led to consider the question as to how far the observed data must be extended in order to obtain a determination of $\Delta(\rho)$ which will satisfy moderate demands at least.

To answer this it will be necessary to determine: (*a*) how the precision of m_0 (prob. err. ρ_{m_0}) depends on the precision of the counts N_m; (*b*) how the precision of the median M depends on the precision of m_0.

---

[1] Since the total number of stars resulting from the two solutions differs enormously, it might seem worth trying to decide between them by measuring the total light of the cloud. To test the feasibility of such a plan I compared the values of the total light and found that the amounts involved in (27) and (28) are as 1.000 to 1.025. Practically, therefore, the idea is worthless, at least with the photometric means at present available.

As for (*a*): Consider first the somewhat easier question: What will be the accuracy of m_0 obtained from observed values of N_m for three, favorably chosen, equidistant magnitudes

$$m_1-\sigma; \qquad m_1; \qquad m_1+\sigma\,? \tag{33}$$

For brevity call this the simplified problem. Since a rough estimate is all that is wanted, assume that the probable error of log N_m for these three magnitudes is constant and equal to r. When this simplified problem has been solved it will be easy to estimate roughly the accuracy obtainable by the use of all values of N_m.

In the Appendix is derived the formula which shows the relation between the probable errors ρ_{m_0} and r, namely,

$$\rho_{m_0}=\pm\frac{r\sqrt{2}}{4\sigma^2 c\mu}\sqrt{12(m_0-m_1)^2+\sigma^2}\,. \tag{34}$$

The value of r is readily determined by means of the deviations of the observed log N_m from a close-fitting curve. In the case of the star-cloud the number of observed stars brighter than 12 is rather small. This being the case, consider only the Mount Wilson observations in Table VIII, which show a better agreement *inter se* than the Harvard values. For the best-fitting curve I used a third solution, intermediate between (27) and (28), for which the constants are:

$$\begin{array}{llll} a=-10.019 & h=-6.038 & & \\ b=+1.570 & k=+4.289 & (\text{area}=40'\times 40') & (35) \\ c=-0.0294 & l=-1.165 & & \\ m_0=26.73 & \text{Median } M=63{,}800 \text{ parsecs} & & \end{array}$$

The value found for the probable error is $r=\pm 0.008$. Experience gained in other cases indicates that this value is accidentally low and that a better one would be

$$r=\pm 0.012. \tag{36}$$

For the fundamental magnitudes we choose the values (see Table VIII)

$$12.18, \qquad 14.52, \qquad 16.86, \tag{37}$$

then for the computation of ρ_{m_0} by (34)

$$\left.\begin{array}{ll} m_1 = 14.52 & \sigma = 2.34 \\ m_0 = 26.73 & c = -0.0294 \end{array}\right\} \qquad (38)$$

which lead to

$$\rho_{m_0} = \pm 2.6 \text{ mag.} \qquad (39)$$

Starting from solution (27) instead of (35), we find nearly the same result.

As for (*b*): We learn how the accuracy of the median M depends on that of m_0 by comparing the values of m_0 and M in the three solutions (27), (28), (35). Adding a fourth solution, we have altogether

$$\left.\begin{array}{cccc} m_0 & & \log M & \\ \hline 25.68 & & 4.592 & \\ & 1.05 & & 0.213 \\ 26.73 & & 4.805 & \\ & 1.05 & & 0.203 \\ 27.78 & & 5.008 & \\ & 2.10 & & 0.426 \\ 29.88 & & 5.434 & \end{array}\right\} \qquad (40)$$

Evidently log M changes very nearly proportionally to m_0. In fact

$$\log M = -0.544 + 0.2\, m_0. \qquad (41)$$

Therefore

$$\rho_{\log M} = 0.2\, \rho_{m_0}; \qquad (42)$$

consequently, adopting the values (39),

$$\rho_{\log M} = \pm 0.520 \text{ (simplified problem).} \qquad (43)$$

It is now easy to see what can be gained by future extension of the counts to fainter magnitudes.

Table IX shows how the probable errors of both m_0 and log M change with increase in the limiting magnitude. In accordance with what precedes, we adopted ±2.6 as the probable error of m_0 in the case of the limiting magnitude 17.5, i.e., for limiting $N_m = 17.0$. The remaining probable errors were then computed by formulae (34) and (42). The results are summarized in Table IX.

Hence, if we were able to photograph the stars to magnitude 20.5 (visual scale), the probable error of log M in the simplified problem would be diminished to 0.176, which, roughly speaking, is equivalent to a probable error of about 42 per cent in the numerical value of M. I believe the error would be further lowered to less than 30 per cent, if the calculation of m_0 were based on all the available data for N_m instead of merely on those for the three fundamental magnitudes used for the simplified problem.

From an absolute point of view this precision may not seem to be very satisfactory. Still if it can be reached—and at present this is hardly subject to doubt—an enormous step in advance will have been made, for it means that the median parallax will have been determined with a probable error of about $\pm 0.''00001$.

TABLE IX

PROBABLE ERROR OF m_0 AND LOG M (SIMPLIFIED PROBLEM)

Limiting Magnitude	Fund. Mags.			m_1	σ	m_0 Sol. (35)	ρ_{m_0}	$\rho_{\log M}$
17.5	12.0	14.5	17.0	14.5	2.5	26.73	2.60	0.520
18.5	12.0	15.0	18.0	15.0	3.0		1.73	.346
19.5	12.0	15.5	19.0	15.5	3.5		1.21	.242
20.5	12.0	16.0	20.0	16.0	4.0		0.88	.176
21.5	12.0	16.5	21.0	16.5	4.5		0.66	.132
22.5	12.0	17.0	22.0	17.0	5.0		0.51	0.102

Moreover, the accuracy can be greatly increased by taking several fields instead of but one.

The preceding considerations on the structure of one small region may of course be applied without change to the investigation of larger parts of the sky, for instance, to the Selected Areas for different zones of galactic latitude. The only difference is that we will obtain a much greater accuracy.

From the data of *Groningen Publication*, No. 27, Table V, we have found the values of m_0 for different galactic latitudes (β) entered in the second column of Table X. The other quantities in the table were obtained in the manner already explained.

In this table the probable errors are given, in the simplified problem, as they would be found for a single region containing the same number of stars as that in the case treated above. In conformity with (39), ρ_{m_0} for $\beta = 0$ and lim. mag. 17.5 was put at ±2.6. For the rest the computation was made with the aid of (34) and (42).

As was to be expected, the values of m_0 run somewhat irregularly. We therefore adopted the smoothed values of the third column, which, with the exception of that for $\beta = 20°$, represent the observations practically as well as those of the second column.

By using for each region all the material instead of the three values of N_m of the simplified problem, the probable error will be diminished. If further for each galactic latitude we take the greater part or all of the Selected Areas, the probable error will again be much reduced. Finally, a further important reduction can be made by taking into account the very extensive data for the brighter stars. In order to see more clearly what can be gained, some estimate must be made as to the effect of all these reducing causes. The matter certainly deserves careful study. We have not yet made such a study, but estimate provisionally that it should be easy to reduce the probable errors of Table X to one-third.[1]

TABLE X

PROBABLE ERROR IN SIMPLIFIED PROBLEM FOR A REGION 15′×15′

β	m_0	Adopted m_0	c	Lim. Mag. 17.5		Lim. Mag. 20.5	
				ρ_{m_0}	$\rho_{\log M}$	ρ_{m_0}	$\rho_{\log M}$
0°........	26.7	26.7	−0.0294	±2.6	±0.52	±0.88	±0.176
20........	27.2	25.6	− .0297	2.4	.48	.79	.158
40........	23.1	22.9	− .0333	1.6	.32	.51	.102
60........	19.1	19.9	− .0407	0.85	.17	.24	.048
80........	18.2	17.9	−0.0486	±0.45	±0.09	±0.11	±0.022

For clearness we may express roughly the accuracy corresponding to this assumption in percentages of the whole. The results for the median M are thus found to be as given in Table XI.

Granting the suppositions made in III*a*, and that we are in possession of data for N_m to magnitude 20.5, we conclude that we can obtain what may be considered a good insight into the distribution of the stars in all galactic latitudes. With data to 17.5 alone (Mount Wilson Survey) this can be said only for the regions of high galactic latitude. Below latitude 40° the solution becomes rapidly less reliable.

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[1] The comparison of the values m_0 in the second column of Table X with the smoothed values in the third leads at once to a precision of about this amount, notwithstanding the fact that the data of *Groningen Publication*, No. 27, extend but little beyond magnitude 14.5 visual.

Remark.—Meanwhile we need an extension of star-counts for regions near the galactic pole quite as much and probably more than for the lower latitudes, for just these observations of N_m in the higher latitudes will lead to a judgment on the general validity or non-validity of the analytical form (2). This follows because of the lower value of m_0 (Table X). Suppose for instance that we obtain counts to apparent magnitude 20.5. In high latitudes the curve for N_m will then become known considerably beyond the maximum, that is, for a very considerable fraction of the whole, and a fraction very much greater than that covered in the lower latitudes. In case formula (2) is confirmed for this large part of the curve in the region of the galactic pole, we will use the same analytical form with much greater confidence for the lower latitudes also.

TABLE XI

PROBABLE ERROR OF MEDIAN *M* IN PERCENTAGE OF ITS AMOUNT

Gal. Lat.	Limiting Magnitude	
	17.5	20.5
0°	41	13
20	38	12
40	25	8
60	13	4
80	7	1.5

III*b*. We know by observation that as far as magnitude 14 the values of N_m are well represented by (2); for the stars fainter than 14 we know nothing of the sort.[1] What results will it be possible to obtain from this knowledge?

Schwarzschild's solution, which is based on the supposition that *all* values of N_m fall on the curve (2), led to formula (8) which, but for the observation errors, would yield a knowledge of $\Delta(\rho)$ for *all* distances. This solution is not now generally applicable and we must look for another. Such a solution, based on quite different principles, which I will call solution K, has already been given in *Groningen Publication*, No. 11, and has also been applied in *Contribution* No. 188. It has there been carried through for galactic latitude 40° to 90°, and shows the very gratifying fact, which we had no right to expect beforehand, that, but for small distances (where Schwarzschild's solution is confessedly wrong) it agrees wonderfully well with the Schwarzschild solution. With the exception referred to, therefore, solution (7) represents all the data, those of *Contribution* No. 88 as well as the values of N_m in a way that leaves hardly anything to be desired.

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[1] We take the limit $m=14$ because it represents what at present is really well known. The reasoning, of course, holds for any other limit.

Therefore, notwithstanding what has just been said, the Schwarzschild solution (8) must still be considered the most plausible to be obtained with the aid of the data at present available. This must be owing to the fact that further values of N_m, particularly those for the most influential magnitudes 15, 16 . . . . , do not deviate very much from the curve (2). Still this does not prove that formula (8) gives even an approximate solution for $\Delta(\rho)$ at very great distances. We see, for instance, from Table II, that according to (8) there are, in high galactic latitudes, hardly any stars beyond $\log \rho = 4.2$. As a consequence thereof, suppose that between $\log \rho = 4.0$ and $\log \rho = 4.2$, the density is a hundred fold the value given by (8); the total numbers of stars N_{14} will be changed by about 1 per cent only. For the brighter magnitudes the change would be still smaller. Evidently, therefore, nothing can be concluded from our data about the densities at such great distances. They may or may not agree with formula (8); we have no means of judging. The matter would be different if we had good counts (N_m) for very much fainter stars. The question thus presents itself: To what distance can the densities computed by (8) be accepted with some confidence? Or to be more precise: To what distance can we trust the values of $\Delta(\rho)$, found by (8), to be correct within 25 and 40 per cent, respectively, of their values?

In considering this question we will *assume* that what was found in *Contribution* No. 188 for galactic latitudes 40° to 90° will also be found for other galactic latitudes, viz., complete agreement of solution K with solution (8). It seems to me very probable that such will turn out to be the case. As solution K is independent of the values of N_m for $m>14$ we may thus accept formula (8) for $\log \rho \leqq 2.8$[1] (excepting of course the small distances).

The question under consideration comes to this: How much can we change the densities (8) for $\log \rho > 2.8$, without spoiling the representation of the values of N_m to $m=14$? Suppose that, for galactic latitudes 40° to 90°, we multiply the densities for

$$\log \rho = 2.8 \text{ to } 3.0 \text{ by } F_1,$$
$$\log \rho = 3.0 \text{ to } 3.2 \text{ by } F_2, \text{ etc.}$$

Then the numbers in Table II (second part) for $\log \rho = 2.8$ to 3.0 will all be multiplied by F_1; those of the following line by F_2, etc.

The total number N_8 (see Table II) will thus be changed to

$$N_8(0.9190+0.0569F_1+0.0188F_2+0.0044F_3+0.0008F_4+0.0001F_5) \quad (44)$$

and, in order that the N_8 may still be represented, the sum must still equal N_8; or better, the expression in parentheses must equal 1.00. We also find similar expressions for magnitudes 9, 10, ... 14.

The question thus becomes: What is the smallest distance for which we can take for any one of the factors F, a value as high as 1.25 or as low as 0.75 (or, respectively, 1.40 and 0.60) without making it impossible to determine the remaining factors in such a way that the seven equations of condition are still bearably well satisfied. For "bearably well satisfied" we have used the condition that the range in the residuals should not exceed 10 per cent.

Suppose, therefore, having assumed the constant F for distance R to be 1.25, that we solve our equations of condition and obtain indeed a "bearable" set of residuals. Will this prove that for distance R a density deviating 25 per cent from (8) is admissible? By no means. For as a rule the values found for the other constants will be impossible or inadmissible because they lead to negative values of $\Delta(\rho)$ or to values running too wildly for the regularly increasing distances.

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[1] In *Contribution* No. 188 this is the limit to which solution K extends.

In order to obtain an acceptable solution we must evidently introduce the condition that the values of F shall run smoothly for regularly increasing distances. This condition is pretty vague and introduces a certain degree of arbitrariness, but I think that there is really no serious objection to it. Moderate differences in the form of the curve will probably change the results little as long as we leave two or three constants indeterminate, to be found in a way best satisfying the conditions of the problem.

Before stating the form adopted for the factors F, consider the general course they must follow. From the value 1.00 for $\log \rho = 2.8$ they must either increase or decrease. Consider only the first case, because by satisfying this we more than satisfy the other. The initial increase cannot indefinitely continue because if all the factors are greater than 1.00, the total numbers N_m will all be increased. The values of N_m cannot well be represented, therefore, unless values of $F > 1.00$ are compensated by later factors < 1.00. The factors must therefore pass through a maximum, after which decrease sets in and they soon become 1.00. We assume that there is but one maximum. Consider now the various curves representing various hypotheses for the factors F. Take more curves for which the value of the maximum is respectively $F_{max} = 1.25$, $F_{max} = 1.30$, $F_{max} = 1.35$, ... and let R, R_1, R_2, ... be the values of the distance for which $F = 1.25$.

Then we think that generally

$$R < R_1 < R_2 \ldots .$$

That is, the factor $F = 1.25$ will be reached sooner by that curve for which $F = 1.25$ is a maximum.[1]

Since we wish to find the smallest distance at which F can become equal to 1.25, we consider only the latter case, that is, find the minimum distance, R, at which F can reach 1.25 as a maximal value, without being led to an "unbearable" representation of N_m.

This being premised, we now give the mathematical form adopted for F:

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[1] We have not tried to find the demonstration. We have been content to verify its truth further on by means of the data in Table X.

$$\left.\begin{array}{l}\text{A. For the case, maximum factor} = 1.25\\ \quad\text{Before the maximum, } F = 0.96 + 0.29\, e^{-h^2x^2}\\ \quad\text{After the maximum, } F = 1.25\, e^{-p^2x^2}\end{array}\right\} \tag{45}$$

$$\left.\begin{array}{l}\text{B. For the case, maximum factor} = 1.40\\ \quad\text{Before the maximum, } F = 0.96 + 0.44\, e^{-h^2x^2}\\ \quad\text{After the maximum, } F = 1.40\, e^{-p^2x^2}\end{array}\right\} \tag{46}$$

in both of which

$$x = \log \rho - \log R. \tag{47}$$

There remain, therefore, the three constants R, h, p, to be determined in such a way that R corresponds to the smallest distance for which the values of N_m can be represented with residuals not exceeding 10 per cent. This form for the factors F gives perfect smoothness throughout. It is that which a priori seems most plausible. Beyond the maximum, $\Delta(\rho)$ is evidently again represented by a form like (8), which represented the densities for the smaller distances with such perfection.

It does not, however, start at $\log \rho = 2.8$ with the initial value 1.00, as premised. The reason is that the small discontinuity thus introduced at $\log \rho = 2.8$ is of no real importance, while the representation of N_m is appreciably better.

Substituting the expressions for F into (44) and equating the result to the successive values of N_m we obtained a number of equations of condition.

In solving them we generally began by assuming two values for R, for each of which these equations were solved for h and p in such a way that practically the best possible representation of N_m(to $m = 14$) was obtained. The range of the residuals was then found for both cases. The value of R for which the range is 10 per cent could then usually be found by interpolation. If not, the computation was repeated for a third value of R. In this way were found all the values entered in Table XII.

In all cases, with the exception of the third for $m = 14$, the range of the residuals in N_m is nearly 10 per cent. In the exceptional case it is only 6.5 per cent. In fact, in this case, the knowledge of N_m to $m=14$ takes us little beyond what was reached by solution *K*.

From Table XII we may be pretty sure, now that we know N_m with some precision to $m=14$, that in the Milky Way there will be in the densities as computed by (8), no error of 25 per cent up to $\log \rho=3.1$ and no error of 40 per cent up to $\log \rho=3.2$. As soon as we have reliable knowledge of N_m to $m=20$ these limits will rise to $\log \rho=3.8$ and 4.0, respectively. We further learn from the table that by extending our knowledge of N_m from $m=14$ to $m=20$ we will extend the limits to which we can reliably determine the structure of the system:

TABLE XII

LIMITS OF RELIABILITY OF $\Delta(\rho)$ BY (8)

GAL. LAT.	N_m KNOWN TO $m=14$				GAL. LAT.	N_m KNOWN TO $m=20$			
	F_{max}	$\log R$	h	p		F_{max}.	$\log R$	h	p
0°	1.25	3.1	7.0	1.66	0°	1.25	3.8	4.0	1.66
0.	1.40	3.2	6.0	2.70	0.	1.40	4.0	3.7	3.6
40 to 90. . .	1.25	2.9	. . . . . .	2.20	40 to 90. . . .	1.25	3.4	4.0	2.05
40 to 90. . .	1.40	2.9	. . . . . .	3.0	40 to 90. . . .	1.40	3.5	4.0	3.73

$$\begin{array}{ll}\text{In the Milky Way} & \left.\begin{array}{l}5.6 \text{ fold } (\log=0.75)\\ 3.5 \text{ fold } (\log=0.55)\end{array}\right\}\\ \text{In Gal. Lat. } 40^\circ \text{ to } 90^\circ & \end{array}\qquad(48)$$

In the case of the Mount Wilson Survey, which will yield N_m to $m=17$, the gain will be about half this amount.

III*c*. If finally, having extended the counts N_m to some higher magnitude, we find that the values of N_m no longer fit an error-curve like (2), then evidently formula (8) will no longer hold for the densities. In this case we must fall back on formula (7). We will thus find different values for the densities $\Delta(\rho)$. We may certainly expect, however, that, as long as these values do not deviate very widely indeed from the densities furnished by such a formula as (8), our considerations on the limits of reliability will still practically hold. For the present, therefore, we may safely rest content with the results just obtained.

POINT IV. EXTENSION OF THE LIMITS OF TABLE XII

There is, we think, reason to assume that the limits shown in Table XII give rather too unfavorable an impression of the real reliability. The reason is that the deviation, having once reached 25 (or 40) per cent, will grow no further, or at least not much further before it begins to diminish. This it will continue to do until it becomes *zero* and begins to increase again and, possibly, go on increasing indefinitely (on the other side). That is to say, beyond the limits assigned in Table XII the deviation from (8) for a certain time will still be mostly very moderate or small. Still much more important than this is a fact which will appear in a further publication, by one of us, in which a formula is derived theoretically which in the last analysis is an expression of the differential quotient $d\Delta(\rho)/d\rho$ as a function of ρ and $\Delta(\rho)$. The integration of this differential equation, if the equation could be accepted as definitely proved, would give $\Delta(\rho)$as a function of ρ for all distances. As it is, the equation is verified by observation as far as observation can fairly be relied on.

We thus have not only reliable values of $\Delta(\rho)$ up to the limits of Table XII, but also to these same limits, reliable values of $d\Delta(\rho)/d\rho$. This proves that we have a right to extend the use of the formula which yielded $\Delta(\rho)$ considerably beyond the limits of Table XII.

There is a third consideration which will enable us to extend the limits of Table XII with some probability of reliable results. In the publication just alluded to it will be shown that the equidensity surfaces are at least approximately similar ellipsoids. If we suppose that this form holds somewhat beyond the limits for which we have fair certainty, then the limits of Table XII will again be extended, at least for the lower galactic latitudes.

The three considerations (of which the second is the most important) together make us believe, that as soon as we have good data for N_m down to magnitude 17, which will be very shortly, we shall be able to find fairly reliable values of the $\Delta(\rho)$ for the entire region near the center for which $\Delta(\rho)$ exceeds one hundredth of density near sun, that is, for the whole of the space represented in Fig. 2, *Contribution* No. 188.

CONCLUDING REMARK

The conclusions arrived at in this paper may perhaps best be seen in their true bearing by comparing the limits within which we can find fairly reliable results for the structure of the stellar system by using different sets of data. Of course Table XIII claims no other merit than that of furnishing the means of a rough insight into what can be obtained at present or in the very near future.

TABLE XIII

LIMITS IN PARSECS WITHIN WHICH THE STRUCTURE OF THE STELLAR SYSTEM CAN BE FOUND

Method	Galactic Latitude		
	0°–20°	40°–90°	0°–90°
1. Direct parallax determination			50
2. Parallactic motion, now well known, to $m=10$	320	240	300
3. The same, to $m=13$	(830)	(610)	(720)
4. Parallactic motion, stars to $m=10$ and $\mu=0''.01$	400	320	360
5. The same together with N_m (the latter to $m=14$)	1600	800	1200
6. The same, N_m to $m=17$	(4000)	(1600)	(2800)
7. Extension according to Point IV, N_m to $m=14$	3000	1000	2000
8. The same, N_m to $m=17$	(8000)	1700	$\Delta=\Delta_0/100$

NOTES TO TABLE XIII

The limits in parentheses have not yet been reached but may be attained in two or three years.

1. The magnificent results lately obtained in direct parallax determinations by means of long-focus instruments show probable errors of the order of $0''.01$. Even if there were demonstrably no systematic errors left, a probable error of $0''.01$ would imply that we cannot derive from them the structure of the system beyond the limit of the table. Since the absence of systematic error cannot be claimed, the limit of the table is probably rather high.

2, 4. The table gives simply the mean distances corresponding to the mean parallaxes in *Groningen Publication*, No. 29, and in a paper not yet published.

3. Simple repetition of the parallax plates obtained at the Mount Wilson Observatory some five or more years after their first exposure, would give excellent values for the parallactic motion of the stars of magnitude 13. Similar repetition of the parallax work done in America on Boss stars by other observatories with long-focus instruments would also contribute to an important degree.

5, 6. From Table XII of the present paper.

7, 8. According to Point IV of present paper. In the last column of No. 8, $\Delta=\Delta_0/100$ means that for all latitudes the densities are fairly reliable for the whole of the space where the density is greater than 0.01 of that near the sun (compare Fig. 2, *Contribution* No. 188).

All the results will be greatly strengthened as soon as we know N_m to $m = 20$. At the same time this will enable us to make good estimates for still greater distances.

APPENDIX

POINT III*a*. ON THE SIMPLIFIED PROBLEM

Given, r = probable error of each of the quantities $\log N_1$, $\log N_2$, $\log N_3$, required, ρ_{m_0} = probable error of the constant m_0.

By formulae (31) and (2) of the text

$$\log N_m = \mu a - 2\mu m_0 c m + \mu c m^2 \qquad (\mu = \text{mod.}). \tag{a}$$

Hence

$$\left.\begin{aligned} \log N_1 &= \mu a - 2\mu m_0 c(m_1 - \sigma) + \mu c(m_1 - \sigma)^2 \\ \log N_2 &= \mu a - 2\mu m_0 c m_1 + \mu c m_1^2 \\ \log N_3 &= \mu a - 2\mu m_0 c(m_1 + \sigma) + \mu c(m_1 + \sigma)^2 \end{aligned}\right\} \tag{b}$$

from which for m_0

$$m_0 - m_1 = \frac{(H+1)\sigma}{2(H-1)} \tag{c}$$

in which

$$H = \frac{\log N_2 - \log N_1}{\log N_3 - \log N_2}. \tag{d}$$

Consequently

$$\rho_{m_0} = \frac{\sigma}{(H-1)^2}\rho_H \tag{e}$$

in which the probable error ρ_H must be found from (d). Since we assume

$$\text{Probable errors of } N_1,\ N_2, \text{ and } N_3 \text{ each} = r, \tag{f}$$

we obtain as in the text

$$\rho_{m_0} = \pm \frac{r\sqrt{2}}{4\sigma^2 c\mu}\sqrt{12(m_0 - m_1)^2 + \sigma^2}. \tag{g}$$

Leiden, Holland
August 1921

COMPARISON STARS FOR NOVA PERSEI, No. 2
A CORRECTION

Father Hagen has called my attention to an error in the identification of one of the comparison stars for Nova Persei, No. 2, whose magnitudes are given in *Mt. Wilson Contr.*, No. 192; *Astrophysical Journal*, **52**, 183, 1920. The star in question is No. 57 of his sequence. An uncertainty in the position and brightness was noted during the measurement of the photographs, but was overlooked when the MS was prepared several months later.

The confusion arose from the fact that the declination coordinates of stars 57 and 62 ($+12\overset{\prime}{.}6$ and $+11\overset{\prime}{.}5$, respectively) are interchanged in the Chart and Catalogue of Father Hagen. The star actually observed here is close to the catalogue position of No. 57, but is much fainter than that object. Neither No. 57 nor 62 has been measured on the Mount Wilson photographs; No. 57, in fact, is just outside the limit of distance from the center of the plate usually adopted. The visual magnitudes on the Mount Wilson scale are readily found, however, by reducing Father Hagen's values to that system. The results are

No. 57 $12\overset{\mathrm{M}}{.}7$; No. 62 $12\overset{\mathrm{M}}{.}8$

F. H. SEARES

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THE QUANTUM THEORY OF BAND SPECTRA AND ITS APPLICATION TO THE DETERMINATION OF TEMPERATURE

By RAYMOND T. BIRGE

ABSTRACT

Structure and theory of spectrum bands.—(1) *A summary of recent work* by Fortrat, Heurlinger, Kratzer, and others is given. It is now known that bands in the visible have the same structure as infra-red bands, each starting from a line of zero intensity and increasing in intensity symmetrically in both directions to maxima, the separation of the lines in one branch meanwhile decreasing until it becomes zero and then negative, the branch reaching a head. It is shown that Kemble's formula, $m^2_{\text{max}} = 0.699T/\Delta\nu_0$, for the number of the line of maximum intensity, m_{max}, as a function of absolute temperature and limiting separation of successive lines for $m = +1$, a formula based on the statistical distribution of angular velocity given by the quantum theory, may be expected to give fairly accurate values of T though it neglects some factors. (2) *Evidence* is presented to show *that Kemble's formula holds* not only for infra-red bands but also for the CN band λ 3883.

Cyanogen band λ 3883.—(1) *Position of maximum intensity as function of temperature.* From a study of spectrograms made by Strutt and Fowler and by King, m_{max} is found to vary from 9±2 for light from a discharge tube containing active nitrogen at about 100°C, to 23±2 for light from a furnace at 2500° to 2700°C, while the values corresponding to these temperatures as computed from Kemble's formula are 8.4, 22.9, and 23.7 respectively. This agreement justifies us in using the formula to determine the approximate temperature in the case of arcs, the sun, and similar sources which emit or absorb these bands. (2) *The intensity distribution in the A series from a 13-ampere arc* is symmetrical and corresponds to the empirical formula $\log I = 1.5 \log m - c_2 m^{1.5} + \text{const.}$ (3) *The relative intensity of the A, B, C, and D series* seems to be *independent of the temperature.*

Spectroscopic determination of temperature, using CN bands.—From the position of maximum intensity on spectrograms made by the author, the *temperature of a 13-ampere arc* was computed to be 4500°C±300, while from King's spectrograms *the temperature of a 4-ampere arc* comes out 3900°C±300. By comparing solar spectrograms taken by St. John with the arc spectrograms the *temperature of the reversing layer of the sun* comes out 4000°C±500.

INTRODUCTION

King[1] has recently obtained spectrograms of the λ 3883 CN band[2] in the electric furnace, at temperatures of 2300° C., 2500° C., and 2700° C. It seemed to the author that these spectrograms furnished valuable material for the quantitative testing of a number of points in connection with the quantum theory of band spectra, and Dr. King has kindly loaned for this purpose all his original plates. He furnished also an absorption spectrogram of the λ 3883 band, taken in the furnace with the plug at about 3500° C. All of these plates were taken in the third order of the 30-foot plane grating of the Pasadena laboratory, and have a dispersion of 1.73 mm per A. The author is indebted also to Dr. St. John, of the Mount Wilson Observatory, for a solar spectrogram of this band, having a dispersion of 1.76 mm per A. I have, in addition, a series of spectrograms of the λ 3883 band, as emitted by the carbon arc, taken in the third order of the 21-foot concave grating of the University of Wisconsin (dispersion 1.13 mm per A). Enlargements, of varying intensity, were made of all of these plates, on a uniform scale of 1.05 cm per A (the scale of King's published enlargement). These and numerous direct contact prints have furnished the chief material for a study of the intensity relations in this band. Additional quantitative wave-length measurements were made from the original negatives, using the eight-inch Gaertner comparator for which the author is indebted to the Rumford Committee of the American Academy of Sciences. Two preliminary reports on this work have been made to the American Physical Society.[3] The present paper is concerned mainly with the results submitted in the later report.

The recent important development of the quantum theory of band spectra, as applied particularly to diatomic molecules, was initiated independently by Heurlinger[4] and by Lenz.[5] Further developments, of use in the present investigation, have been made by Heurlinger[1] and by Kratzer.[2] In the case of ordinary bands, such as the λ 3883 band, the empirical formulation of the experimental material is due chiefly to Heurlinger,[3] who elaborated an idea first suggested by Fortrat.[4] The results are briefly as follows.

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[1] *Astrophysical Journal*, **53**, 161, 1921.

[2] Theoretical considerations indicate that this band is due to nitrogen, but its origin is still a matter of dispute (see Barratt, *Proceedings Royal Society* [A], **98**, 40, 1920, and Holst and Oosterhuis, *K. Akad. Amsterdam, Proc.*, **23**, 5, 727–728, 1921).

[3] Berkeley, August, 1921 (Abstract, *Physical Review*, **18**, 319, 1921) and Toronto, December, 1921.

[4] *Physikalische Zeitschrift*, **20**, 188, 1919.

[5] *Verhandlungen der deutschen physikalischen Gesellschaft*, **21**, 632, 1919.

There is no essential difference in the structure of near infra-red bands, such as the halogen acid bands investigated by Imes[5] and by Colby and Meyer,[6] and many ordinary bands in the visible and ultra-violet, such as the λ 3883 CN band, investigated in great detail by Uhler and Patterson,[7] Birge,[8] Heurlinger,[9] Bachem,[10] and many previous workers. Each true "band" consists of one "series" having two similar branches. The common origin of these two branches is a line of zero intensity, and is labeled $m=0$. From this point the positive branch (Heurlinger's "*P*" branch) runs toward the longer wave-lengths and successive lines are designated $m=+1$, $+2$, etc. The negative branch (Heurlinger's "*R*" branch) runs in the opposite direction and the lines are designated $m=-1$, -2, etc. Successive lines are not equally spaced but crowd closer together in one direction, until the series presently reaches a turning-point, where the separation of successive lines has become zero, and then turns back on itself. This turning point is what has always been known as the "head" of a band. Either the positive or the negative branch may thus invert. In the former case, illustrated by the λ 3883 CN band, the band is said to be degraded toward the violet, while in the latter case, illustrated by the halogen acid infra-red bands, the band is said to be degraded toward the red.

The two types of bands, ultra-violet and infra-red, do not appear similar in structure, due mainly to the fact that the infra-red bands have not been followed to the "head" because of the rapid decrease in intensity of the lines. But this in turn is due merely to the low temperature of the absorbing molecules, in the experimental conditions employed. Theoretically, each line of a band series corresponds to a different change of molecular rotation. The entire series corresponds to the same change of atomic vibratory motion for the near infra-red bands, or to a simultaneous change in atomic vibratory and electronic rotatory motion, in the case of visible and ultra-violet bands. The exact relation between the labeling of the lines (value of m) and the number of quantum units of angular momentum was in doubt until the work of Kratzer (*loc. cit.*). His conclusions, which seem to the author to be entirely valid, are that the missing line ($m=0$) indicates, in emission, the transition from one to zero units of angular momentum ($1\rightarrow 0$). In absorption it indicates the reverse transition $0\rightarrow 1$. The positive branch always indicates, in emission, an increase in momentum, while the negative indicates a decrease. The relations are reversed in the case of absorption. The value of m, for any line, thus indicates in the case of absorption, the initial number of units of angular momentum possessed by the molecule, while in the case of emission it indicates the final number. Thus in emission $m=+60$ indicates the transition $59\rightarrow 60$, $m=-59$ indicates the transition $60\rightarrow 59$.

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[1] *Zeitschrift für Physik*, 1, 82, 1920.

[2] *Ibid.*, 3, 289, 1920.

[3] Dissertation, Lund, 1918.

[4] Thesis, Paris, 1914.

[5] *Astrophysical Journal*, 50, 251, 1919.

[6] *Ibid*, 53, 300, 1921.

[7] *Ibid.*, 42, 434, 1915.

[8] *Ibid.*, 46, 85, 1917; *Physical Review*, 11, 136, 1918.

[9] Dissertation, Lund, 1918.

[10] *Zeitschrift für Physik*, 3, 372, 1920.

In the case of all visible and ultra-violet bands, it has been customary to consider a series as starting at the point for which $\Delta\nu=0$, where a strong head generally appears, and the lines are numbered from this point. The numbers thus obtained will be indicated, in the present article, by m', and in the case of the λ 3883 band, the actual numbers for the A series are those used by Uhler and Patterson (*loc. cit.*). The idea that a series has two branches, centering at a point other than the "head" of the band, is due to Fortrat (*loc. cit.*) who found that the lines showing irregularities of position (the "perturbations") could thus be arranged in pairs at $\pm m$. Heurlinger found that by taking for this new origin the point of minimum intensity, similar perturbations occurred for the lines[1] $+m$ and $-(m-1)$. The reason for this discrepancy of one unit does not seem to have been explained, but the explanation follows immediately from Kratzer's hypothesis and is an important verification of the truth of this hypothesis. Thus the lines +60 and −59 show the same perturbation, and as previously noted these two lines indicate transitions between the same two rotational energy states (59 and 60) but in opposite directions. There is evidently an irregularity in this transition which affects the emitted and absorbed frequency. The experimental material at hand has furnished new data on the quantitative relations, in the case of perturbations, but this matter is reserved for a separate article. Suffice it to say here that Bachem's results (*loc. cit.*) for the perturbations, from magnetic effects, are independently confirmed by King's furnace plates, while the similarity of corresponding perturbations is even greater than has been supposed.

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[1] Heurlinger in his dissertation had not yet theoretically interpreted the two branches. He seems to have adopted the policy of calling the branch which returned on itself the negative branch, and gave the lines of this branch negative values of m. This is theoretically correct for all bands degraded to the red, but is incorrect for all

Heurlinger found that by taking the origin of a band series at the point of minimum intensity, the two "branches" of the series show qualitatively a similar intensity-curve. Each branch begins at zero intensity, rises to a maximum, and then decreases. Heurlinger also called attention to the fact that the intensity distribution in at least one branch changed with physical conditions and could be interpreted as a change due to temperature. In the case of near infra-red bands, each branch changes with temperature, the point of maximum intensity moving toward the higher m's, as the temperature increases. The main purpose of the present investigation has been to make a quantitative study of this matter for the λ 3883 band.

The theoretical material on this point is as follows. The intensity[1] of an absorption line (in band spectra) depends on three factors, i.e.,

$$I=(E_{m'}-E_m)B^{m}_{m'}\,a_m \qquad (1)$$

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bands degraded to the violet. In agreement with most other investigators such as Bachem (*loc. cit.*) the author in his August, 1921, report used the original Heurlinger nomenclature for the λ 3883 band. But in the present paper the theoretically correct nomenclature is used, thus reversing the sign of m, and the designation of the two branches. In the old nomenclature, the rule for equal perturbations reads $+m$ and $-(m+1)$.

[1] Heurlinger, *Zeitschrift für Physik*, 1, 82, 1920.

where the transition is from m to m' units of angular momentum. The first factor (energy change) is proportional to the emitted frequency and changes by only 6 per cent during the course of the longest (A) series of the λ 3883 band. The second factor gives the probability of a transition from m to m', under the influence of absorbed radiation of the proper frequency. Little is known concerning this factor except that, from experimental data, it changes rather slowly in value with m, if m is not too small. It is also probable that

$$B^{m}_{m+1}=B^{m}_{m-1}.$$

The third factor, a_m, gives the number of molecules, at any instant, possessing m units of angular momentum. A statistical distribution of angular momentum is to be expected, on either classical or quantum theory, and such a distribution was computed first by Kemble,[1] who applied it to the 4.7 μ CO infra-red band, on the assumption that the intensity depended on this one factor alone. Kemble, as a matter of fact, worked out the distribution of angular velocity, and accordingly the results hold for angular momentum only in case $J=$constant, where J refers to the initial moment of inertia, in the case of absorption. This does not exclude a possible change in the moment of inertia (from J to J') during the transition, due presumably to the change in vibration and electronic configuration. What is required is that J and J' be independent of the amount of molecular rotational energy. If $J=J'$ the lines of a series would have constant frequency difference. If J = constant, and $J'=$another constant, then Deslandres' parabolic law should be true.[2] But Deslandres' formula is probably not accurately true for any known band,[3] including infra-red bands, and this indicates that J (and presumably J') both vary slowly with m. No attempt has yet been made to interpret the variation quantitatively in terms of J and J'.

Thus Kemble's formula is not a true expression for a_m, but probably approximates it very closely. The value of m corresponding to the line of maximum intensity is given, in Kemble's formula, by

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[1] *Physical Review* (2), 8, 689, 1916.

[2] See Sommerfeld's *Atombau* (second edition), pp. 550–566, for a summary of the work underlying these conclusions.

[3] Birge, *Physical Review* (2), 13, 360, 1919.

$$m_{\max}=\frac{2\pi}{h}\sqrt{kJT}. \qquad (2)$$

where h = Planck's constant (6.55×10^{-27} erg-sec.), k = gas constant per molecule (1.372×10^{-16} erg/degree) and T = absolute temperature. If J is constant, its value can be accurately obtained from the limiting separation of successive lines ($\Delta\nu$) for $m=+1$, in which case

$$\Delta\nu_0=h/4\pi^2J.^1$$

Hence it is possible to compute the temperature of the absorbing molecules, from the value of m for the line of maximum intensity, in either branch of a band series. Heurlinger (dissertation, p. 65) obtained this result, but considered that from experimental evidence only qualitative values of temperature were to be expected, in the case of ordinary band spectra. It will, however, be shown in this article that from recent experimental evidence the various approximations made must have relatively little effect, so that a close estimate of temperature is possible. Since in the case of discontinuous emission and absorption by vapors, no other method of quantitative temperature determination is as yet known, any results, even if approximate, should be of value in both terrestrial and astronomical work.

A formula similar to (1) applies to the case of emission, but the explanation of the expected value of the various factors is more involved. It therefore seems simpler to use the fact of the known

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[1] In Sommerfeld's *Atombau* (p. 555, 2d edition) J' denotes the true moment of inertia (i.e., that corresponding to the initial state in the case of absorption, or to the final state in the case of emission). Sommerfeld's equations then indicate that J' is to be obtained from the limiting $\Delta\nu$ for $m=0$. But, as used by Sommerfeld, m refers to the initial amount of angular momentum, in the case of emission, and as Kratzer has shown, the line $m=0$ is then not the missing line, but the first line of the positive branch. Hence the true moment of inertia (designated J in this article) is to be obtained directly from the value of $\Delta\nu$ corresponding to the first line to the red of the missing line (designated here as $m=+1$) while the value of $\Delta\nu$ corresponding to the missing line itself gives J', the moment of inertia of the molecule in an "energized" condition. Kratzer himself used an analytic method for obtaining the correct value of $\Delta\nu$, for the halogen acids, but an equally accurate value can be obtained readily from a simple graph of $\Delta\nu$ and m, using the value of $\Delta\nu$ corresponding to $m=+1$.

similarity of absorption and emission spectra, and to express all results, for either emission or absorption, in terms of the absorption formula.

EXPERIMENTAL DATA AND RESULTS

The λ 3883 CN band consists actually of five overlapping bands, each formed by one complete series. The positive branches of each of the five series (called the A, B, C, D, and E series) return on themselves, and thus cause the five heads of this band. Each series is a doublet series, with variable doublet separation (Δn), but corresponding terms ($\pm m$) in the two branches have the same value of Δn, another criterion for the division into branches. The value of Δn is zero for $m=0$. From here Δn increases to about 0.037 A at $m=-28$ (the first resolvable doublet) and to a maximum of about 0.075 A at $m=-75$. It then diminishes toward zero, the last resolvable doublet being $m=-126$. It is probable that the entire $\Delta n:m$ curve is symmetrical with respect to $m=75$, and can be considered as practically linear in the region of the unresolved doublets. This latter assumption was checked by the measured width of the unresolved doublets, after correcting for variation due to intensity.

Because of the blending of lines of the various series, the strong A series presents the best material for study. It has, however, been possible to secure enough data on the other series to indicate quite definitely that the results herein presented for the A series apply to each of the other series. The A series has its origin ($m=0$) at λ 3875.198, this being the theoretical position of the missing line ($m'=-28$) of the A_1 "singlet" series.[1] The positive branch proceeds toward the λ 3883 head as a series of apparently single (A_1) lines, but returns as a series of doublets (A_2). Actually, as noted, the A_1 lines are increasing in width and from analogy with the negative branch, the first resolvable doublet should be just at the λ 3883 head ($m=+28$). It is this apparent discrepancy in the two portions of the positive branch which so long hindered a true realization of the facts. The details concerning this and other matters connected with the frequency relations in the λ 3883 band are reserved for a separate paper.

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[1] If, beginning at the λ 3883 head, the lines of the A_1 series are labeled by negative values of m', while the adjoining A_2 doublet to the red is given by an equal positive value, then $m=m'+28$, always. The missing B_1 line is $m'=-29$. Hence the transformation for the B series is $m=m'+29$, while for the C series it is $m=m'+30$.

A quantitative curve of the intensity distribution of the negative branch of the A series (consisting entirely of A_1 lines) has been obtained from the series of exposures (all on the same plate) taken with a 13 ampere 170 volt carbon arc at the University of Wisconsin. The usable exposures vary from 40 minutes to 15 seconds in length. The results were obtained by finding lines of equal photographic blackening, assuming the "Reciprocity Law" (It = constant, for equal blackening). This law is known to be only approximate, and in even the modified Schwarzschild formula IT^p = constant, it appears that p can have a considerable range of values, depending on many different circumstances.[1] However, under the conditions employed by the author, it is believed that the error over the entire intensity-curve will be less than possibly 50 per cent, while for the restricted portion of the curve which is of chief value, the error should be quite negligible.

It was found that by using on the prints only lines of rather low intensity, the error of a single comparison was about 10 per cent, and this could be greatly reduced by averaging many independent comparisons. The prints were cut up, and the lines to be compared brought into actual contact. One of the inherent difficulties is due to the fact that resolved doublets have frequently to be compared with unresolved doublets. This was partly overcome by making prints on medium contrast (Azo) paper, so that none of the doublets were resolved. On maximum contrast Azo (used for both prints and enlargements) the definition is equal to that on the original negatives, while with proper exposure, the intensity variations are far more evident to the eye.

In this work it is essential that there be used only lines quite free from all other lines. While it is frequently evident that lines are "blends," it is not so evident in many other cases, and certain knowledge on this point results only from a complete analysis of all of the series of the band, i.e., a computation of the "theoretical" position of every line of every series. The author, as a result of his work on this band (*loc. cit.*) has such data for all of the stronger series, and the only series or portions of series still not analyzed are of such low intensity that they can scarcely produce any appreciable error. This data, most of which is still unpublished, has been of invaluable aid in this entire investigation. A very few theoretically "free" lines show irregular intensity, but in most cases this is evidently due to the effect of extraneous lines (line spectrum of carbon?) of which there are quite a number scattered throughout the band.

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[1] See Odencrants, *Zeitschrift für wissenschaftliche Photographie*, **16**, 111, 125, 189, 1916, for complete review of work to date, and Helmick, *Physical Review*, **17**, 135, 1921, for the latest results.

The variation of intensity throughout the observed A series lines is about seven hundred fold. The Kemble formula does not fit the intensity-curve obtained. Thus if the theoretical and experimental curves are made to coincide at the point of maximum intensity ($m = -30.4$) the experimental value at the last known line at $m = -137$ (4.5 times 30.4) is 5.5 times too large, and this evidently exceeds the possible experimental errors. As far as the chance experimental errors of the blackening comparisons are concerned, it is believed that the curve is accurate to about 10 per cent, except for very small values of m (0–5) where the error may be much greater. Other possible sources of error are (1) change of sensitiveness with wave-length of the plate, (2) uneven illumination of the printing paper, (3) uneven development of the plate or paper. It is thought that none of these causes separately would produce errors much in excess of 10 per cent. It is therefore believed that the foregoing discrepancy is due to the various approximations inherent in the use of the Kemble formula for the distribution of intensity, i.e., the neglect of the first two factors in equation (1), and the assumption of constant J.

It is convenient to have a purely empirical equation for the experimental curve, and such an expression is found to be

$$I = c_1 m^{\frac{1}{2}} e^{-c_2 m^{\frac{1}{2}}}. \qquad (3)$$

The corresponding Kemble formula is

$$I = c_1 m e^{-c_2 m^2} \qquad (4)$$

Equation (3) fits the experimental curve to within about 10 per cent from $m=5$ to 137, taking $m_{\max.}=30.4$.

The only previous quantitative work on the distribution of intensity in this band is that of Haferkamp,[1] where, however, only the region λ 3883 to λ 3865 was studied. The quantitative portion of this work was based purely on the equality of the three lines of certain A_1A_2 triplets. At these points ($m'=15$ and 35, according to Haferkamp) the A_1 and A_2 series are taken to be of equal intensity. Actually, however, since the A_1 line is an unresolved doublet, the A_2 term is twice as strong as the A_1. In addition Haferkamp finds an irregularity at $m=-4$. This results from his misinterpretation of the A_2 perturbation at that point. The lines comprising the perturbations are, I believe, regular as regards intensity, and irregular only as regards position. The net result is that Haferkamp's intensity-curves are more or less irregular, and are not symmetrical for the two branches, a fact which has caused a real difficulty in the application of theory.[2]

Now while it is not possible to obtain directly a complete intensity-curve for the positive branch of the A series, enough intensity comparisons of the lines of the two branches can be made to make possible the conclusion that within the rather narrow limits of experimental error the distribution of intensity in the two branches is probably exactly the same. This result seems to the author of considerable importance, for it shows that these CN bands are entirely similar to the infra-red bands in intensity distribution, and thus furnishes a foundation for the use of the quantum theory of band spectra in the quantitative determination of temperature. But while the two branches seem to have intensity-curves of the same shape, the actual intensity of the positive branch averages some 10 to 20 per cent greater than that of the negative. The difference appears to be independent of the wave-length separation of the two lines compared (amounting to 3 to 85 A), and so can scarcely be due to a possible change with wave-length of the sensitiveness of the photographic plate.

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[1] *Zeitschrift für wissenschaftliche Photographie*, 9, 19, 1910.

[2] See Eucken, *Jahrbuch d. Radioaktivität und Elektronik*, 16, 397, 1920.

Equation (1) indicates why the distribution of intensity in the two branches should be at least very closely the same. For in the case of absorption each pair of corresponding lines ($\pm m$), have the same a_m term; theory indicates that the middle factor should have the same value for the lines of a pair (i.e., the transitions $m \rightarrow m-1$ and $m \rightarrow m+1$ are equally probable for absorption); and the first factor differs at the most by 2 per cent (85 A) for the two lines of a pair.

While the experimental results thus indicate that Kemble's formula is only approximate, as applied to the complete intensity distribution in the λ 3883 band, it seems to the author probable from theoretical considerations that equation (2) giving the $m_{\max}$ should be more nearly correct. Let us first see if this equation checks in the case of infra-red bands. E. v. Bahr,[1] using the classical theory, showed that between 17° and 1430° C. the frequency separation of the maxima of the two branches was directly proportional to $\sqrt{T}$, as predicted. On quantum theory it is not the frequency separation, but the separation in terms of number of lines that should vary with $\sqrt{T}$, i.e., $m_{\max}=k\sqrt{T}$. If the lines were equally spaced, as they are approximately, the two viewpoints would coincide. In the work quoted the results to be expected by the two theories differ by less than the experimental error, so that the quantum theory is equally well substantiated.

The best infra-red data is that by Colby and Meyer (*loc. cit.*) for the fundamental HCl band, at room temperature, and at a temperature described as "the tube was heated to a glow which could just be seen in a darkened room." I take this to be about 500° C. It is convenient to express formula (2) in terms of the limiting $\Delta\nu$ for $m=+1$. ($\Delta\nu_0=h/4\pi^2J$.) One thus obtains

$$T=1.431\ \Delta\nu_0 m_{\max}^2 \qquad (5),$$

where $\Delta\nu_0$ is expressed in frequency units. Using the accurate value of $\Delta\nu_0$ (21.30) computed by Kratzer[2] (*loc. cit.*), for $T=300°$ absolute $m_{\max}=3.13$, for 773°, $m_{\max}=5.03$. Colby and Meyer's curves (*op. cit.*, p. 301) indicate that these values are correct, as nearly as can be ascertained.

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[1] *Verhandlungen der deutschen physikalischen Gesellschaft*, 15, 731, 1913.

[2] This value is from Imes's data. Colby and Meyer's data yield the same value, from the $\Delta\nu{:}m$ graph.

Let us turn again to the λ 3883 band. The outstanding feature of King's furnace spectrograms is the redistribution of intensity in each branch of the various series, as compared to the arc exposures.[1] Due presumably to the lower temperature of the furnace, the $m_{\max}$ has a smaller value, and starting at $m=0$ each branch increases very rapidly in intensity. There is a correspondingly rapid falling off of intensity for the larger values of m. Thus in my 13 ampere arc the lines $m=4$ and 83 are approximately equal, while in the 2700° furnace the equality is between $m=4$ and 52. In the 13 ampere arc $m=52$ is 4.5 times as strong as $m=83$. But again theory is verified by the fact that the redistribution of energy is the same for each branch. Thus self-reversal starts in the 2700° furnace at $m=61$ and $1\frac{1}{2}$, in each branch; and in the 2500° spectra at 51 and 3, in each branch, as nearly as can be judged. A direct comparison of $m=\pm 1$, ± 2, and ± 3 seems to indicate a small difference in absolute intensity of the two branches, but probably not as great as that in the case of the arc spectra. The ± 61 and ± 51 equalities would indicate no difference, but this criterion is not a very sensitive one.

The experimental difficulties in determining intensities are much greater, in the case of the furnace spectra, due to the self-reversal of the stronger lines. It has, however, been possible to obtain $m_{\max}$ directly by measurements of the width (a) of the self-reversed portion of the lines, since this width varies rapidly with intensity. King's furnace plates contain an arc comparison (arc of about 4 amperes, 170 volts). The width (e) of the emission line was subtracted from a. This eliminates the variation in a due to changing doublet separation. (In the region used the doublets are still unresolved.) The (a–e) computed was then corrected for the variation in e due to intensity, the amount of this correction being determined by a separate study of the width of single arc emission lines. The resulting corrected values of (a–e) gave a fairly smooth symmetrical curve, with a definite maximum. In the case of arc spectra, a direct determination (method 1) of $m_{\max}$

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[1] See King, *op. cit.*, Plate XX.

is possible, as well as an indirect method (method 2) from pairs of lines of observed equal intensity, using formula (3). The direct comparison is of course virtually the comparison of pairs of lines of equal intensity, but close to $m_{\max}$ so that the arithmetic mean may be used.

One of the first facts noted in this investigation was that King's 4 ampere arc did not show the same intensity distribution as my 13 ampere arc. Thus consider the A_1A_2 "triplet" $m'=33$ (the first to the red of the second head). In the 4 ampere arc all three lines are of practically equal intensity, as shown on King's published enlargement. In the 13 ampere arc the A_1 line is decidedly weaker than the two A_2 lines. As far as can be ascertained, equation (3) applies equally well to the 4 ampere arc. Nothing can be said as to the furnace spectra, since the data are too fragmentary. Likewise no quantative results seem possible, for the 2300° C. furnace plate, due to the weakness of the lines.

The experimental values of $m_{\max}$ are as follows:[1]

		Method 1	Method 2
13 amp. arc, $m_{\max}$	=	30 ± 1	30.4
4 amp. arc, $m_{\max}$	=	28 ± 1	28.2
2700° furnace $m_{\max}$	=	23 ± 2	
2500° furnace $m_{\max}$	=	23 ± 2	

Further material on this band is furnished by its appearance in active nitrogen.[2] Although the published photograph is one with rather small dispersion, I estimate $m_{\max}$ at 9 ± 2. The A series at this point is however contaminated by the B series, and a more accurate value is obtainable from the λ 4216 band, since here the A series is not thus contaminated. It has, moreover, practically the same $\Delta\nu_0$ and therefore its $m_{\max}$ should be the same. The result is $8\pm1\frac{1}{2}$. The peculiar appearance of these bands, in active nitrogen, is in fact due solely to the radical redistribution of intensity in each branch of the various series. Thus in the neighborhood of $m=0$, the usual appearance of the band, under low dispersion, is due almost solely to the A_2 lines, the A_1 lines being very faint. But in active nitrogen, the A_2 series is practically missing, due to the rapid decrease in intensity for higher values of m. These photographs also indicate most beautifully the similar redistribution in the other series of the CN bands. The relative intensity of the A, B, C, and D series, in active nitrogen and in King's furnace plates, seems to be changed but little, if at all.

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[1] Since unconscious bias can easily enter into such determinations, it seems only fair to state that all of the quantitative results here presented were obtained before their quantitative theoretical significance was fully realized, and no change of any kind in the data has since been made. The value of 22 given for the furnace spectra in the August report was a preliminary one, based on a few uncorrected (*a–e*) readings.

[2] Strutt and Fowler, *Proceedings Royal Society* (A), **86**, 1911. Plate opposite p. 117.

The value of $\Delta\nu_0$ for the A series of the λ 3883 band[1] is 3.704. Putting this value in equation (5) we have

$$T=5.30\,m^2_{\max}. \tag{6}$$

From

$$m_{\max}=8\pm 1\tfrac{1}{2},\quad T=70^\circ\ \text{C.}\pm 150^\circ.$$

Strutt considers the temperature in active nitrogen about 100° C. E. P. Lewis thinks this estimate too high. The temperature must be above room temperature but probably very little above. Using the reverse process for the furnace spectrograms, we have

$$T=2700^\circ\ \text{C. gives } m_{\max}=23.7$$
$$T=2500^\circ\ \text{C. gives } m_{\max}=22.9$$

The experimental values were 23 ± 2 in each case. It therefor seems to the author that there is fairly conclusive evidence to the effect that over a considerable range of temperature, formula (2) is capable of giving quantitative results. This formula admittedly involves a number of real approximations, but the errors thus introduced seem to be smaller than the necessary experimental errors involved in determining $m_{\max}$.

It further seems that formula (2) should yield at least very approximate results in the case of higher unknown temperatures, in the case both of absorption and of emission spectra. It will be noted that in the previous data, the HCl band is in absorption, the active nitrogen in emission, and the furnace spectra mostly self-reversed and so a combination of emission and absorption. In postulating a similar intensity distribution for emission and absorption spectra at the same temperature, all that is assumed is that two lines of equal intensity in absorption will be of equal intensity in emission, and that the line having the greatest absorption coefficient will also show the greatest emitted intensity. The actual photometric curve of an absorption spectrum will of course depend on the depth and density of the absorbing layer. The same statement is true, to a lesser degree, for emission, as is indicated by the phenomena of self-reversal.

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[1] Heurlinger's value of c_1 ($=\Delta\nu$ for $m=0$) is 3.84. The value of $\Delta\nu$ for $m=+1$ is obtained from $c_1-2c_2=3.84-2\times 0.068$.

King's absorption plate of the λ 3883 band yields information on this subject. The plate was taken with the plug at 3500° C., and Dr. King estimates the effective temperature of the vapor as in the neighborhood of 3000° C. An accurate comparison of this plate with the furnace spectrograms is difficult, due as usual to the self-reversal in the latter, but the results are as follows. The absorption intensity relations are definitely different from those of the 2500° C. furnace plate, and as nearly as can be ascertained, the same as those of the 2700° C. plate. The difference, if any, is such as to indicate a higher temperature. I, therefore, estimate its temperature as 2700–2900° C. From the known plug temperature, this value cannot be very greatly in error.

Formula (2) can now be applied in two cases to the determination of unknown temperatures, (1) the CN vapors of the electric arc, (2) the CN vapors in the sun's reversing layer.

From the data for $m_{\max}$ already given (method 1), we have for the 13 ampere arc $T=4500°$ C$\pm 300°$, and for the 4 ampere arc $T=3900°$ C$\pm 300°$. It is known that the hottest portion of the arc crater is about 4000° C., but it seems very probable that the vapors are still hotter, since it is the impact of the electrons and ions of these vapors which causes the crater temperature.[1] The author is aware that there is considerable uncertainty as to just what is meant by "temperature" in a case like this, where there is no thermal equilibrium. The same problem enters into many questions concerning solar and stellar temperature. The term "effective temperature" is therefore used, leaving open the question of its relation to temperature, as measured by more ordinary methods.

[1] See Marx, *Handbuch der Radiologie*, 4, 378–390, for summary of work on this subject.

In the case of the solar spectrum of the λ 3883 band, it is impossible to obtain directly the value of $m_{\max}$. The only available criterion is a comparison of relative intensities of adjacent lines in the solar and in the arc spectrograms. The best lines for this purpose are the "triplet" groups in the neighborhood of $m=0$. For the A_1 lines quickly rise in intensity, going in each direction from this point, and presently equal in intensity the adjacent A_2 lines. The point where this equality occurs varies with the temperature. The available data are even more fragmentary than is indicated in Rowland's table,[1] for a detailed comparison of intensities, using enlargements of St. John's solar plate, indicates that many CN lines marked as free by Rowland are actually blends.[2] The only really available triplet groups seem to be those for which $m'=16$, 18, 19, and 26. Of these $m'=19$ (λ 3879.851, λ 3879.796, and λ 3879.716, on Rowland's scale) furnishes the most sensitive criterion. For this group, the furnace spectra and King's absorption plate show A_1 much stronger than the two A_2 lines. King's 4 ampere arc shows the three lines equal. In my 13 ampere arc A_1 is slightly weaker than A_2. In the sun the three lines seem equal. If there is any difference, A_1 is the stronger. Hence the solar temperature is in the neighborhood of the effective temperature of the 4 ampere arc (4000° C.). A study has been made also of a large number of "free" solar CN lines, and the results are in at least qualitative agreement with this conclusion. A quantitative study does not seem possible.

Grebe and Bachem's micro-photometric curves[3] yield similar results.[4] It should be noted, in this connection, that the curves on page 419 are mislabeled. The actual A_1 lines marked by dots in both arc and sun are $m'=25$, 23, and 18, and not 26, 24, and 19 as the printed wave-lengths indicate. In these curves the red is to the right, except in the solar λ 3879.716 curve. The scale is 2 cm = 1 A.

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[1] *Astrophysical Journal*, **1**, 29, 1895.

[2] My first estimate of solar temperature (4700° C.) was partly due to the accidental use of such lines.

[3] *Zeitschrift für Physik*, **2**, 419, 1920.

[4] Grebe and Bachem's λ 3873.504 curves seem to furnish the best material for this test. But this triplet group ($m'=31$) is not usable in the sun, and the published "solar" curve has seemingly no relation to the actual solar spectrum at this point.

The results show that Grebe and Bachem's arc is about the same temperature as my 13 ampere arc, while their solar curves indicate a lower temperature than their arc, but possibly not as low as that of the 4 ampere arc. This temperature of about 4000° C., according to Dr. St. John,[1] is lower than anticipated, but I feel that it is probably correct to about 500° C.

The only other method for such determinations is that of Saha[2] but this method requires a knowledge of the pressure, so that the temperature determinations may be in error by large amounts. It would seem as though the present method, with increasing theoretical knowledge, should be capable of fairly accurate quantitative results. Meanwhile many possible investigations immediately suggest themselves, such as a determination of the relative temperature in sun-spots and in other portions of the solar atmosphere, and the variation of effective arc temperatures with varying physical conditions. In this connection it is to be remembered that in addition to the CN bands, others such as the λ 5165 Swan band[3] are already known to exhibit similar variations with temperature, and may therefore be used in such work.

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[1] Personal communication.

[2] *Zeitschrift für Physik*, **6**, 40, 1921, and contained references.

[3] See King (*loc. cit.*); Konen, *Das Leuchten der Gase und Dampfe*, p. 317; and Heurlinger, Dissertation, pp. 46–55.

THE ARC SPECTRA OF GALLIUM AND INDIUM

By H. S. UHLER and J. W. TANCH

ABSTRACT

Arc spectra of gallium and indium, λ *4511 to* λ *2170*.—*To identify the lines*, spectrograms obtained with the pure elements and with various compounds and mixtures were compared; these were made with a short focus concave grating spectrograph and with a large quartz prism spectrograph. *For the accurate determination of the wave-lengths*, spectrograms were made with a 21-foot concave grating, special precautions being taken to minimize pole effects, to avoid mechanical displacements and to obtain the lines sharp and separated from close lines of other elements. The wave-lengths for air, accurate in most cases to a few thousandths of an angstrom, and the vacuum wave-numbers are given for 23 gallium lines and for 34 indium lines. All the gallium lines and 28 of the indium lines were found to belong to the *two subordinate series*, for each of which from 5 to 8 terms were identified. Preliminary values of the *constants of the Ritz-Paschen formula* for these series were calculated for both elements, but the agreement between observed and calculated wave-numbers is not satisfactory. The Mogendorff-Hicks formula was tested for gallium with even worse results. Four indium lines, previously supposed to be single, were resolved into from three to six components each.

Absorption spectrum of indium in a furnace.—Specially designed furnaces were used but only λλ 4511 and 4102 were obtained.

INTRODUCTION

The present paper largely fulfils the promise made in the following quotation: "As soon as possible the wave-lengths of the gallium lines will be accurately determined by means of a 21-foot grating so that the constants of the series formula may be calculated independently of the analogous data of aluminium, thallium, indium, etc."[1]

The necessity for the investigation is sufficiently stated by H. Kayser[2] in the sentence: "Alles in Allem ist das Spectrum des Ga noch so gut wie unbekannt." The data for indium given in the fifth and sixth volumes of the same classic show that spectroscopic work on this element is also highly desirable.

The bibliography of the subject may be dispensed with by merely referring to Kayser's *Handbuch* and to a more recent paper by F. Paschen and K. Meissner.[3]

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[1] Philip E. Browning and Horace S. Uhler, *American Journal of Science*, **41**, 351, 1916; see also, *ibid.*, **42**, 389, 1916.

[2] *Handbuch der Spectroscopie*, **5**, 461, 1910.

[3] *Annalen der Physik*, **43**, 1223, 1914.

The experimental work was begun before the year 1916 and it was closed in August 1919, but the attempted calculations and preparation for publication were inordinately delayed, first by the war and later by the temporary incapacity of the senior author.

The investigation involved two problems, the first of which consisted in vaporizing carefully purified metals in the arc in order to identify all of the lines radiated respectively by gallium and by indium, and to obtain the approximate wave-lengths of these lines. The second problem involved a more detailed study of the discrete lines and the more accurate determination of the wave-lengths of all the lines, except the very faintest ones.

APPARATUS, MATERIAL, AND METHOD

For the identification of the lines most of the spectrograms were obtained with two instruments: (*a*) a grating spectrograph[1] of one meter radius of curvature designed by the senior author, and (*b*) a large quartz prism spectrograph of the Littrow type made by A. Hilger and owned by the American Brass Company in Waterbury, Connecticut. Most of the films used in the grating apparatus were made either by the Wratten & Wainwright Company or to special order—through the courtesy of Dr. C. E. K. Mees—by the Eastman Company. The plates employed in the prism spectrograph were Wratten and Wainwright backed, red-sensitive plates and also special Schumann plates made and presented by Dr. I. C. Gardner. Unfortunately the plates last mentioned did not bring out any more lines in the ultra-violet than were recorded on the Wratten plates. Since the Schumann plates were of excellent quality, the negative result obtained with them indicated that the lines of high order were not radiated with observable intensity under the experimental conditions of the arcs. The metals were prepared and purified by the senior author, and they were vaporized upon positive electrodes either of Acheson graphite or of copper. This part of the work should be thoroughly reliable, as a large number of spectrograms were also taken with the materials which Browning obtained while studying the chemical separation of gallium and indium.

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[1] *American Journal of Science*, 41, 353, 1916.

For the more accurate determination of the character and wave-lengths of the stronger lines the large grating spectrograph of the Sloane Laboratory was used. The grating was ruled by J. A. Anderson; it has a radius of curvature of about 640 cm (21 ft.) with 590 lines per mm (15,000 per inch). The mounting is according to Rowland's plan and it is characterized by great rigidity. The foundation upon which the whole apparatus rests is of solid masonry and it is situated in the sub-basement of the building. The grating-room was specially designed to maintain constant temperature, a condition that is admirably fulfilled. The spectral lines between λ 4511 and λ 2932 were obtained by using both the second and the third orders, while the range between λ 2932 and λ 2300 was obtained only in the second order. The linear dispersions in the second and third orders are about 1.32 A per mm and 0.87 A per mm, respectively. The grating was very carefully tested to ascertain whether it would admit of the use of the "method of coincidences." This test was necessary because the ultra-violet lines of the third order were determined by interpolation between the standard iron lines of the second order. It was found that the grating fulfilled the requirements well within the limits of experimental error. This result was not unexpected, since it is in full accord with the investigation of Miss J. T. Howell.[1]

In several cases (e.g., Ga λ 2874) the lines under investigation sensibly coincided with iron lines and hence necessitated the use of a semaphore. The base upon which this screen was mounted was screwed to the rigid foundation of the spectrograph, very close to the incidence side of the camera. In order to avoid displacing the plate during exposure, the semaphore was raised or lowered by means of cords passing through the partition between the arc-room and the grating-room.

The plates used with the large grating were Cramer Isochromatic Instantaneous and Seed 26 and 30. Jewell's simple hydrochinone developer was invariably used with these plates.

In order to minimize the pole effect of the iron arc and to comply with the then recognized regulations for international standards, a modified form of the Pfund arc was employed.[2] This arc had a length between 8 mm and 12 mm with a direct current of 5 amperes or less fed by 110 volt mains. Under these conditions it burned steadily as long as desired without restriking. A length of about 1.25 mm at the central portion of the arc was used as the source of illumination of the slit, the light from the remaining portions being screened off by a metal diaphragm which was perforated by a rectangular hole 1.25 mm high and much wider horizontally. This diaphragm was placed about 7 mm from the flame and it was rigidly attached to the base of the lamp. The upper, or negative, electrode was a rod of common carbon, while the positive electrode consisted of pure electrolytic iron kindly supplied by the United States Bureau of Standards. All the other specimens of iron tested contained manganese as an impurity, several of the lines of which nearly coincided with certain gallium and indium lines.

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[1] *Astrophysical Journal*, 39, 230, 1914.

[2] *Ibid.*, 42, 231, 1915, and 46, 150, 1917.

The secondary interferometer standards published by the Bureau of Standards[1] were used for the region between λ 4528 and λ 3233, while for wave-lengths shorter than λ 3233 the earlier data given by Fabry and Buisson were employed.

The original source of the gallium and indium was a leady residue which contained these metals as impurities to the amount of about 0.5 per cent and 2.0 per cent, respectively. This material was usually shaved into small fragments with an ice-plane and then tamped into a deep hole that had been drilled in the axis of a rod of Acheson graphite. When the lines radiated from the positive electrode thus formed were too widely reversed, the leady residue was diluted by stirring little pieces of it in ordinary liquefied lead. After the mixture or solution became homogeneous it was poured into cored carbon or graphite rods. In cases where the lines under investigation were obscured by lead lines, indium hydroxide or indium oxide, containing gallium as an impurity, was used in the cores. In other cases, lead oxide, reduced from lead nitrate which had been obtained as a by-product in the chemical work, was used to advantage. The upper electrode was invariably a rod of common carbon.

The exposure time was usually 5 min. or 10 min., but sometimes an hour or more. In some cases, half the exposure to the iron arc was given, then the full exposure to the material in question, and, finally, the second half of the iron exposure. Check plates were always taken with the iron lines impressed simultaneously with the lines of the rare elements.

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[1] *Bulletin No. 274*, 1916.

RESULTS

Gallium. Experimental.—The numerical data are collected in Table I. The numbers in the first column are merely for reference. The series notation in the second column should be obvious. The third column contains the wave-lengths in air given by Exner and Haschek. The next column presents the new wave-lengths in air obtained by the present authors. The reduction from the fourth column (15° C.) to the fifth (*vacuo*) was based on the formula given on page 722 of *Bulletin No. 327* of the United States Bureau of Standards.

TABLE I

No.	Series	λ in Air		Wave-Number $\nu_0 = \lambda_0^{-1} \times 10^8$	$\Delta\nu_0$	Int.
		E. and H.	U. and T.			
1........	S_1 (2)	4172.06	4172.048	23962.31	826.30	10
2........	S_2 (2)	4033.03	4032.975	24788.61	....	10
3........	D_1' (3)	2944.18	2944.175	33955.47	826.16	6
4........	D_1 (3)	2943.66	2943.639	33961.66	....	6
5........	D_2 (3)	2874.24	2874.240	34781.63	....	6
6........	S_1 (3)	2719.66	2719.664	36758.39	826.23	3
7........	S_2 (3)	2659.84	2659.873	37584.62	....	3
8........	D_1' (4)	....	2500.714	39976.53	826.14	3
9........	D_1 (4)	2500.18	2500.187	39984.96	....	7
10........	D_2 (4)	2450.10	2450.078	40802.67	....	6
11........	S_1 (4)	....	2418.699	41331.96	825.67	2
12........	S_2 (4)	....	2371.325	42157.63	....	2
13........	D_1' (5)	....	2338.596	42747.57	827.12	1
14........	D_1 (5)	....	2338.293	42752.47	....	2
15........	S_1 (5)	....	2297.869	43505.12	826.15	1
16........	D_2 (5)	....	2294.202	43574.69	....	1
17........	$_1D_1'$ (6)	....	2259.227	44249.16	826.24	1
18........	S_2 (5)	....	2255.034	44331.27	....	1
19........	S_1 (6)	....	2236.103	44706.68	823.30	1
20........	D_2 (6)	....	2218.039	45075.40	....	1
21........	$_1D_1'$ (7)	....	2211.753	45198.83	828.	1
22........	S_2 (6)	....	2195.665	45529.98	....	1
23........	D_2 (7)	....	2171.9	46027.	....	1

The differences $\Delta\nu_0$ are printed in the same horizontal line as the other data for the less refrangible of the two components involved. This scheme avoids overlapping of braces and confusion where the subordinate series get out of step.

As is often the case, the column of intensities means but little more than is contained in the following verbal statements. The less refrangible component of each doublet is somewhat stronger than the more refrangible one. The satellites are very much weaker than either of the main components of the same doublet. In any one series the intensities of the lines decrease very rapidly as the order (m) of the line increases. When approximately pure gallium is burned in air all of the lines are widely reversed. For all the strong lines the reversal was suppressed when the metal was caused to radiate as a minute impurity in some mixture. The wave-lengths obtained with the very fine unreversed lines agreed, within the limits of experimental error, with the wave-lengths observed by setting upon the axis of the widely reversed lines.

The lines numbered 1 and 5 received special attention because of their closeness respectively to the iron lines at λ 4172.128 and λ 2874.176. Lines 1 to 5 inclusive were measured on third-order plates, while all of the lines were determined in the second order. When a line was measured in both orders, the difference between the results for each individual line was never greater than 0.004 A, and usually less. Each line was always measured four times in one direction, then the plate was reversed and the line measured four times in the other direction.

Theoretical.—In attempting to represent the series lines of gallium by an empirical equation we began by using the Ritz formula in the following form

$$\nu_m = A - \frac{N}{[m+a+b(A-\nu_m)]^2}, \qquad (1)$$

because it has been so advantageously employed by Paschen and others. In terms of the frequences ν_m, ν_n, ν_r of any three lines of the same single series the above formula easily leads to the condition

$$(\nu_r-\nu_n)\sqrt{\frac{N}{A-\nu_m}}+(\nu_m-\nu_r)\sqrt{\frac{N}{A-\nu_n}}+(\nu_n-\nu_m)\sqrt{\frac{N}{A-\nu_r}} \\ +(r-n)\nu_m+(m-r)\nu_n+(n-m)\nu_r=0, \qquad (2)$$

from which A may be determined by the method of "trial and error." An inspection of formula (2) shows that A is not at all dependent upon the numerical value given to m, say. This follows from the fact that m, n, and r enter equation (2) only as binomial differences. For this reason, and quite regardless of the theoretical considerations of Bohr, Sommerfeld, and others, we have chosen to define m for the first line of a given series as the smallest positive multiple of $1/2$ that will make a numerically less than $1/2$. When A has been determined from relation (2), formula (1) only gives the value of $m+a$ as a sum, but not of m and a separately. These remarks account for the numbers in the parentheses of the column headed "Series" in Table I.

Using R. T. Birge's latest value $N=109677.7$, Dr. E. Klein and the senior author obtained the following data from the wave-numbers of the first three lines of each single series of gallium. For the satellites of the first subordinate series $m=3$, $A'=47491.90$, $a'=-0.2126354$, $b'=+4.366673\times10^{-6}$, and $\lambda_\infty=2105.622$. For the most refrangible components of this series $m=3$, $A=48320.76$, $a=-0.2138487$, $b=+4.434379\times10^{-6}$, and $\lambda_\infty=2069.504$. For the companion of the satellite of the same series $m=3$, $A=47489.91$, $a=-0.2076978$, $b=+4.067868\times10^{-6}$, and $\lambda_\infty=2105.710$. For the least refrangible components of the second subordinate series $m=2$, $A=47553.53$, $a=+0.2138570$, $b=-2.445062\times10^{-6}$, and $\lambda_\infty=2102.893$. Finally, for the most refrangible components of the second subordinate series $m=2$, $A=48378.55$, $a=+0.2141350$, $b=-2.454515\times10^{-6}$, and $\lambda_\infty=2067.032$. These results are unsatisfactory, for the convergence wave-numbers of the least refrangible components of the two subordinate series differ by 61.63 or 2.73 A. The most refrangible components failed to converge to the same place by about the same amount, namely 57.79 or 2.47 A. Formula (1) was considered disproportionately complicated to apply the method of least squares to all of the lines of a single series. However, this method was applied to the Mogendorff-Hicks formula

$$\nu_m=A-\frac{N}{(m+a+b/m)^2}. \tag{3}$$

The equation analogous to relation (2), as derived from equation (3), shows that the value of A depends upon the number assigned to m. By actual trial it was found that the convergence wave-numbers for the subordinate series did not agree even as well as when formula (1) was employed. For the least refrangible components of the second subordinate series, the values of the constants in equation (3) were found, by the method of least squares, to be: $A = 47555.879$, $a = 1.2183276$, $b = -0.06226206$, and $\lambda_\infty = 2102.789$. m was chosen equal to unity for the first line. For the wave-numbers the differences "observed" minus "calculated" were $+0.041$, -0.552, $+0.824$, $+1.063$, and -1.375 or, in terms of the angstrom, -0.007, $+0.041$, -0.051, -0.056, and $+0.069$, respectively. These numbers are written in the order of decreasing wave-length of the lines. The conclusion reached by us is that the number of lines is too small and the perturbations of the lines with respect to a non-inflectional curve are too large to justify pursuing the series constants further.

No attempt was made to determine the wave-lengths of the red and green lines of the principal series of gallium for the following reasons. These lines appeared only on the spectrograms taken with the meter grating and with the large quartz spectrograph when nearly pure metallic gallium was used. They were not recorded with the oxide (which forms very promptly in the arc in air) or with the leady residue, or with any of the compounds or mixtures which contained gallium as an impurity. Whenever obtained, these lines were extremely broad and unsymmetrical. Consequently it was not feasible to make exposures with such valuable material with the large grating as mounted. Moreover, the investigation of the red region was in full sway at the Bureau of Standards, where all of the conditions are most favorable for obtaining the wave-lengths in question. The only earlier, fairly accurate, measurements of the principal series doublets seem to be those of Exner and Haschek which are quoted by Paschen and Meissner.[1]

Indium. Experimental.—Most of the investigations of the radiations of indium have been confined to the spark spectrum, with which we are not directly concerned in the present paper. The lines of the principal series have been very carefully determined by Paschen and Meissner.[2] The results of our study of the arc spectrum in air are given in Table II, the notation being the same as in the preceding table for gallium. Lines 1 to 22 inclusive were obtained with the large grating, while the remaining ones were recorded only with quartz spectrographs. Line number 10 was close to the iron lines at λ 2753.668 and λ 2754.034, and line number 19 to the iron line at λ 2399.339, hence the semaphore screen was used in photographing these indium lines. Special interest attaches to the four lines numbered 6, 8, 9, and 22, respectively. These lines are not single, but are made up of groups of fine components the number of which is indicated by the subscript following the representative wave-

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[1] *Annalen der Physik*, **43**, 1223, 1914.

[2] *Ibid.*

TABLE II

No.	Series	λ in Air		Wave-Number $\nu_0 = \frac{10^8}{\lambda_0}$ cm^{-1}	Δν	Int.
		E. and H.	U. and T.			
1	S_1 (2.5)	4511.38	4511.310	22160.05	2212.83	10
2	S_2 (2.5)	4101.80	4101.764	24372.88		10
3	D_1' (3)	3258.54	3258.565	30679.34	2212.85	4
4	D_1 (3)	3256.08	3256.089	30703.17		8
5	D_2 (3)	3039.34	3039.356	32892.19		8
6		2957.02	2957.012$_4$	33808.33	2212.51	6
7	S_1 (3.5)	2932.64	2932.633	34089.03	2212.49	5
8		2836.90	2836.911$_6$	35239.04		8
9		2775.36	2775.355$_3$	36020.84		4
10	S_2 (3.5)	2753.89	2753.889	36301.52		4
11	D_1' (4)	2713.91	2713.932	36835.87	2212.53	3
12	D_1 (4)	2710.28	2710.264	36885.49		5
13	S_1 (4.5)	2601.81	2601.756	38424.13	2212.73	2
14	D_2 (4)	2560.15	2560.157	39048.40		4
15	D_1' (5)		2522.854	39625.34	2212.90	3
16	D_1 (5)	2521.32	2521.371	39649.04		6
17	S_1 (5.5)	2468.12	2468.012	40506.23	2212.21	5
18	S_2 (4.5)	2460.16	2460.079	40636.86		3
19	S_1 (6.5)		2399.189	41668.05	2211.9	3
20	D_2 (5)		2389.543	41838.24		3
21	S_2 (5.5)	2340.27	2340.191	42718.44		3
22		2306.09	2306.073$_3$	43350.39		4
23	S_2 (6.5)		2278.3	43879.9		2
24	S_2 (7.5)		2240.9	44611.9		1
25	D_2 (8)		2229.8	44833.5		1
26	S_2 (8.5)		2219.4	45044.9		1
27	D_2 (9)		2210.9	45236.3		1
28	S_2 (9.5)		2202.5	45389.2		1
29	D_2 (10)		2198.2	45477.9		1
30	S_2 (10.5)		2192.0	45606.0		1
31	D_2 (11)		2189.0	45669.7		1
32	D_2 (12)		2181.7	45822.9		1
33			2177.7	45906.6		1
34			2171.0	46039.0		1

length given in Table II. Lines 6, 8, and 9 were not obtained by Kayser and Runge,[1] but they are recorded by Exner and Haschek[1] as *single* lines with the respective characterizations 2 *R*, 3 *R*, and 1. Line number 22 is also given as single by both pairs of authors. To minimize the possibility of drawing erroneous conclusions we made a special study of these four lines and found that (*a*) each group displayed the same characteristics whether the exposure had been long or short, the lines faint or intense, (*b*) the character of the lines was independent of the material containing the indium—whether it had been the leady residue or its products, indium oxide or hydroxide, or lead oxide, and (*c*) the lines invariably appeared when indium was present in the arc. Line number 8 suggested a small portion of a band spectrum. It could not be interpreted as a triplet, each of whose components was reversed, because all of the six components were sharp and symmetrical with no suggestion of winglike structure. The four lines are probably "combination lines," since they do not fit into the series directly. The difference in wave-number for lines 6 and 9 (when treated as unresolved) is 2212.51, which is the characteristic value for the series lines of indium. The wave-lengths of the resolved components are given in Table III.

TABLE III

Reference No.	λ	Reference No.	λ
6	2957.094 2957.047 2956.994 2956.735	9	2775.408 2775.355 2775.292
8	2837.039 2836.998 2836.957 2836.910 2836.856 2836.793	22	2306.147 2306.082 2306.029

In Table II the lines numbered 24 to 34 were always accompanied by so much intermediate radiation as to make them appear as absorption lines in a continuous background, rather than as ordinary emission lines with reversed axes. Nevertheless the serious attempts made by Dr. E. Klein and the senior author to obtain the absorption spectrum of indium vapor, when heated in specially designed furnaces, failed to give anything more than the two strongest lines numbered 1 and 2.

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[1] Kayser's *Handbuch*, 5, 584.

Theoretical.—The time-consuming difficulties which presented themselves when the attempt was made to determine the various constants for gallium have caused us to defer a similar numerical investigation for indium until some later occasion. Moreover, as stated earlier, the presentation in published form of the experimental data has already been delayed far too long. Accordingly the following tentative constants must suffice.

Taking $m=2.5$, 3.5, ... for the first three components of the second subordinate series it was found that, for formula (1),

$$\begin{aligned} A_1 &= 44449.3 \\ A_2 &= 46662.0 \\ a_1 = a_2 &= -0.215623 \\ b_1 = b_2 &= -2.96938 \times 10^{-6}. \end{aligned}$$

Assuming the same values for A_1 and A_2, and letting $m=3$, 4, ... the remaining parameters for the satellite and the most refrangible component of the first subordinate series have the approximate values

$$\begin{aligned} a_1 = a_2 &= -0.237740 \\ b_1 = b_2 &= 4.35554 \times 10^{-6}. \end{aligned}$$

In conclusion the authors desire to make public acknowledgment of their debt of gratitude to Dr. I. C. Gardner for his Schumann plates, to Mr. W. H. Bassett, technical assistant and metallurgist of the American Brass Company, for placing at the disposal of the senior author all of the spectroscopic facilities of the research laboratory, and to Mr. C. H. Davis for his untiring and invaluable aid in obtaining the spectrograms with the large quartz prism apparatus.

SLOANE LABORATORY
YALE UNIVERSITY
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FIRST ATTEMPT AT A THEORY OF THE ARRANGEMENT AND MOTION OF THE SIDEREAL SYSTEM[1]

By J. C. KAPTEYN[2]

ABSTRACT

First attempt at a general theory of the distribution of masses, forces, and velocities in the stellar system.—(1) *Distribution of stars.* Observations are fairly well represented, at least up to galactic lat. 70°, if we *assume that the equidensity surfaces are similar ellipsoids of revolution*, with axial ratio 5.1, and this enables us to compute quite readily (2) *the gravitational acceleration at various points due to such a system*, by summing up the effects of each of ten ellipsoidal shells, in terms of the acceleration due to the average star at a distance of a parsec. The *total number of stars* is taken as 47.4×10^9. (3) *Random and rotational velocities.* The nature of the equidensity surfaces is such that the stellar system cannot be in a steady state unless there is a general rotational motion around the galactic polar axis, in addition to a random motion analogous to the thermal agitation of a gas. In the neighborhood of the axis, however, there is no rotation, and the behavior is assumed to be like that of a gas at uniform temperature, but with a gravitational acceleration ($G\eta$) decreasing with the distance ρ. Therefore the density Δ is assumed to obey the barometric law: $G\eta = -\bar{u}^2(\delta\Delta/\delta\rho)/\Delta$; and taking the mean random velocity $\bar{u}$ as 10.3 km/sec., the author finds that (4) *the mean mass of the stars* decreases from 2.2 (sun = 1) for shell II to 1.4 for shell X (the outer shell), the average being close to 1.6, which is the value independently found for the average mass of both components of visual binaries. In the galactic plane the resultant acceleration—gravitational minus centrifugal—is again put equal to $-\bar{u}^2(\delta\Delta/\delta\rho)/\Delta$, $\bar{u}$ is taken to be constant and the average mass is assumed to decrease from shell to shell as in the direction of the pole. The angular velocities then come out such as to make the linear rotational velocities about constant and equal to 19.5 km/sec. beyond the third shell. If now we suppose that part of the stars are rotating one way and part the other, the relative velocity being 39 km/sec., we have a quantitative explanation of the phenomenon of star-streaming, where the relative velocity is also in the plane of the Milky Way and about 40 km/sec. It is incidentally suggested that when the theory is perfected it may be possible to determine *the amount of dark matter* from its gravitational effect. (5) The *chief defects of the theory* are: That the equidensity surfaces assumed do not agree with the actual surfaces, which tend to become spherical for the shorter distances; that the *position of the center of the system* is not the sun, as assumed, but is probably located at a point some 650 parsecs away in the direction galactic long. 77°, lat. −3°; that the average mass of the stars was assumed to be the same in all shells in deriving the formula for the variation of $G\eta$ with ρ on the basis of which the variation of average mass from shell to shell and the constancy of the rotational velocity were derived—hence either the assumption or the conclusions are wrong; and that no distinction has been made between stars of different types.

1. *Equidensity surfaces supposed to be similar ellipsoids.*—In *Mount Wilson Contribution* No. 188[3] a provisional derivation was given of the star-density in the stellar system. The question was there raised whether the inflection appearing near the pole in the

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[1] *Contributions from the Mount Wilson Observatory*, No. 230.

[2] Research Associate of the Mount Wilson Observatory.

[3] *Astrophysical Journal*, **52**, 23, 1920.

equidensity surfaces for small densities is real or not. I have since found that these inflections can be avoided without doing very serious violence to the results of observation. If this is done, the equidensity surfaces become approximately ellipsoids, and not only that, but the data can be represented without exceeding the possible limits of observation error, by assuming the equidensity surfaces to be *concentric*, *similar revolution ellipsoids*, *similarly situated.*

2. *Elements of the ellipsoids.*—Taking as unit of star-density that in the neighborhood of the sun, the adopted axes of the ellipsoids, which will be referred to as ellipsoids I, II, ... X and which correspond to the values (Δ being the density) $\log \Delta + 10 = 9.8, 9.6,$ . . . . 8.0, are as shown in Table I. The A-axis is directed toward the galactic Pole, the B-axis lies in the plane of the Milky Way.

TABLE I

EQUIDENSITY ELLIPSOIDS

Ellipsoid	Log Δ	A	B	B/A
		parsecs	parsecs	
I.	9.80−10	118	602	5.102
II.	9.60	198	1010	5.102
III.	9.40	296	1510	5.102
IV.	9.20	413	2106	5.102
V.	9.00	553	2820	5.102
VI.	8.80	717	3656	5.102
VII.	8.60	902	4600	5.102
VIII.	8.40	1114	5675	5.102
IX.	8.20	1365	6960	5.102
X.	8.00	1660	8465	5.102

For the Milky Way and for the direction toward the Pole this table yields densities which are fairly well represented, for $\rho > 150$ parsecs, by the formulae,

$$\log \Delta = -2.135 + 2.368 \log \rho - 0.593 (\log \rho)^2 \quad \text{(M.W.)}, \tag{1}$$

$$\log \Delta = -5.356 + 4.890 \log \rho - 1.200 (\log \rho)^2 \quad \text{(Pole)}, \tag{1a}$$

A section of the equidensity-ellipsoids through the sun (which has been assumed to be the center of the system) at right angles to the plane of the Milky Way is shown in Figure 1.

The agreement of the densities furnished by Table I with those of *Contribution* No. 188 is fairly good for all galactic latitudes up to 65° or 70°. For still higher latitudes it may perhaps still be called tolerable. At least the deviations hardly exceed what would be produced by an error of 0.1 mag. in the photometric scale for these regions.

In the present paper I have substituted these ellipsoids for the surfaces derived directly from observation in *Contribution* No. 188, *not* because I think they are nearer the truth, but simply because they are so enormously more convenient for further computation.

My aim in the present paper is simply to get hold of some approximate information about the real structure and motion of the system, and quantitative accuracy has been considered of secondary importance as long as we may hope that the main features are not affected. I trust that this hope will not be disappointed, notwithstanding the many defects—defects that will be duly pointed out—which still attach to the present treatment.

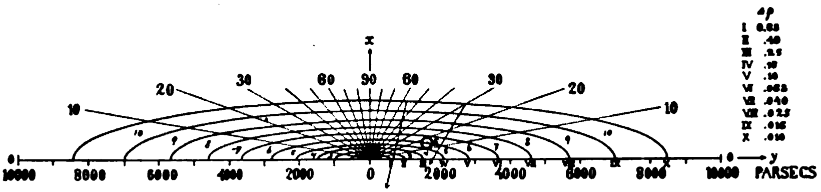

FIG. 1

3. *Advantage of the adoption of the ellipsoids.*—The form of the equidensity surfaces thus adopted has the advantage that it calls attention to the possibility of determining with some precision the gravitational attraction of the *whole* of the stellar system on any point inside ellipsoid X, while at the same time it renders the computation of that attraction a relatively easy matter.

In another paper[1] van Rhijn and I have tried to show that, as soon as we possess good counts of stars for each interval of magnitude down to apparent magnitude 17 (visual), we shall know with some tolerable approximation the density of the whole region covered by Figure 1, that is, of the whole extent of the stellar system for which the density exceeds one-hundredth of that in the neighborhood of the sun.

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[1] *Mt. Wilson Contr.*, No. 229; *Astrophysical Journal*, **55**, 242, 1922.

In the near future such counts will be available. They will be furnished by the Mount Wilson "Catalogue of the Selected Areas" (from $\delta = -15°$ to $\delta = +90°$), the discussion of which is in the hands of Seares. A few provisional counts make it probable that this work will in the main confirm the elements used for Table I and Figure 1. I will assume, therefore, that even now the densities are sufficiently well known for the whole of ellipsoid X.

The advantage just alluded to is a consequence of the well-known property that the attraction of an ellipsoidal shell of constant density, bounded by two similar and similarly situated ellipsoids, on an internal point is *zero*. For it is evident by this property that, if in all that part of the system which lies outside ellipsoid X—for which part accurate data are still wanting—the arrangement in similar ellipsoids also holds, the attraction of this outside domain on a point inside ellipsoid X would be *zero*. And as the distribution of density inside ellipsoid X is known, the possibility of computing the attraction of the total system on a point inside of X becomes evident. If on the contrary the same arrangement does not hold outside ellipsoid X, it still seems highly probable a priori that any change in the form of the equidensity surfaces must be gradual, that is, the equidensity surfaces in the neighborhood of X will diverge little from similar ellipsoids, and the greater changes will begin to appear only at more considerable distances. For the consecutive shells, therefore, the attraction on an internal point will begin by being very small, both on account of the near approach to similarity of these shells and their small density and greater distance from the attracted point. For more distant shells the first circumstance will probably diminish in importance with increasing distance, while, on the contrary, the second becomes more and more important. On the whole, therefore, the attraction of all of that part of the system which lies outside X will be small, and its neglect will presumably not prevent us from obtaining fairly exact ideas about the total forces.

4. *Computation of the gravitational forces.*—In ellipsoid I, which for brevity I will call shell 1, and in each of the shells 2, 3, ... 10, between the surfaces of ellipsoids I, II, ... X, the density varies between limiting values which are in the ratio of 1 to 1.585.

In what follows I will assume for each shell a constant average density.

The computation of the gravitational forces has been carried through for:

(1) 10 points in galactic latitude 0° lying in the surfaces of ellipsoids I, II, ... X; these points have been designated by I, 0°; II, 0°; . . . . X, 0°.

(2) 5 points in galactic latitude 30°, situated on the surfaces II, IV, VI, VIII, X; denoted by II, 30°; IV, 30°; . . . . X, 30°.

(3) Similarly for the 5 points II, 57°.1; IV, 57°.1; . . . . X, 57°.1. (57°.1 = average latitude between 40° and 90°.)

(4) The 10 points I, 90°; II, 90°; . . . . X, 90°.

As a unit of attraction I have used the attraction on each other of two stars of average mass separated by a distance of 1 parsec.

I first computed the attraction of the full ellipsoids I, II, ... X on the points specified above, on the supposition that they are of a constant density such that every cubic parsec contains a single star. The formulae for this computation are given in the Appendix.

The attraction of the full ellipsoids having been found, simple subtraction gives the attraction of the separate shells 1, 2, ... 10, all supposed to have the density corresponding to one star per cubic parsec. The actual attraction of the shells was obtained by multiplying these results by the number of stars per cubic parsec contained in each shell. For the average densities, expressed in terms of the density in the neighborhood of the sun, I adopted the values corresponding to the logarithms 9.9, 9.7, 9.5, ... 8.1, each minus 10, multiplied by 0.0451, which according to *Contribution* No. 188 (12) is the number of stars per cubic parsec near the sun; this gives the numbers in Table II.

TABLE II

AVERAGE NUMBER OF STARS PER CUBIC PARSEC

Shell	No. Stars	Shell	No. Stars
1	0.0358	6	0.00358
2	.0226	7	.00226
3	.0143	8	.00143
4	.00900	9	.000900
5	0.00568	10	0.000568

Having found the separate attractions, the components of the total attractions parallel to the axes can at once be determined by noting that the attraction of any shell on an internal point is *zero*, and further, by neglecting the attraction of that part of the system outside of ellipsoid X on a point inside this ellipsoid. Instead of the components I have entered in Table III the total forces G and the angles that these forces make with the X-axis. These are of course found by the formulae

$$X=G\cos\phi \qquad Y=G\sin\phi. \tag{2}$$

In addition, I have given the values of $\phi-\psi$, ψ being the angle which the normal to the ellipsoid through the attracted point makes with the X-axis; ψ is determined by

$$\tan\psi=\frac{A^2}{B^2}\frac{\beta}{a} \tag{2a}$$

The angle $\phi-\psi$ is of course the inclination of the direction of the force to the normal.

The interpretation of this table—of the first entry, for example—is: Attraction of the whole stellar system on a body in the point I, 0° is a force equal to the attraction of 33.19 stars of average mass at the distance of 1 parsec from that same body. This force makes an angle of 90° with the X-axis, and an angle of 0° with the normal to the equidensity ellipsoid through the attracted point.

5. *Analytical representation of G for galactic latitudes* 0° *and* 90°.—In trying to represent the force G by an analytical formula, I started from the consideration that, as the density is constant near the center, the attraction must be nearly proportional to the distance ρ for very small values of ρ; further, that for distances very great as compared with the dimensions of the stellar system, the attraction must be practically the same as it would be were the mass of the whole system concentrated in the center. For these latter distances, therefore, G must be proportional to $1/\rho^2$.

The following easily managed formula satisfies both conditions:

$$G=\frac{A\rho}{1+B\rho+C\rho^2+D\rho^3}. \tag{3}$$

TABLE III

Total Attractions of the Whole System

Attr. Point	Log Δ	Co-ordinates Attr. Point		G	ϕ	$\phi-\psi$	Attr. Point	Log Δ	Co-ordinates Attr. Point		G	ϕ	$\phi-\psi$
		α	β						α	β			
		parsecs	parsecs						parsecs	parsecs			
I, 0°	9.80−10	0	602	33.19	90°	0°	I, 90°	9.80−10	118	0	40.06	0°	0°
II, 0	9.60	0	1010	37.76	90	0	II, 90	9.60	198	0	54.26	0	0
III, 0	9.40	0	1510	37.95	90	0	III, 90	9.40	296	0	62.45	0	0
IV, 0	9.20	0	2106	35.71	90	0	IV, 90	9.20	413	0	65.95	0	0
V, 0	9.00	0	2820	32.09	90	0	V, 90	9.00	553	0	66.12	0	0
VI, 0	8.80	0	3657	27.71	90	0	VI, 90	8.80	717	0	64.00	0	0
VII, 0	8.60	0	4600	23.67	90	0	VII, 90	8.60	902	0	60.63	0	0
VIII, 0	8.40	0	5675	20.12	90	0	VIII, 90	8.40	1114	0	56.41	0	0
IX, 0	8.20	0	6960	16.59	90	0	IX, 90	8.20	1365	0	51.56	0	0
X, 0	8.00	0	8465	13.53	90	0	X, 90	8.00	1660	0	46.52	0	0
II, 30°	9.60	187	325	52.34	18° 6′	14°18′	II, 57°.1....	9.60	196	127	53.99	6°54′	5°29′
IV, 30	9.20	391	677	59.26	22 6	18 18	IV, 57.1....	9.20	410	265	64.86	8 49	7 24
VI, 30	8.80	679	1176	53.97	25 8	21 20	VI, 57.1....	8.80	711	460	62.10	10 33	9 8
VIII, 30	8.40	1055	1827	44.64	28 2	24 14	VIII, 57.1....	8.40	1105	715	53.91	12 5	10 40
X, 30	8.00	1572	2723	34.46	30 54	27 6	X, 57.1....	8.00	1647	1066	43.85	13 42	12 17

In this formula A/D evidently equals the total number of stars, N, in the stellar system.

A rough estimate of this number can be made by assuming that formula (15) of *Contribution* No. 188, viz.,

$$N_m = e^{a+bm+cm^2}, \qquad (4)$$

is true for all values of m. Integration over m gives

$$N = \frac{\sqrt{\pi}}{\sqrt{-c}} e^{a-\frac{b^2}{4c}}. \qquad (5)$$

For the values of a, b, c best satisfying the numbers in *Groningen Publications*, No. 27, Table V, which gives number of stars per square degree, I find the following:

Gal. Lat.	a	b	c	Number of Square Degrees
0° to ±20°	−10.564	+1.5985	−0.0276	14110
±20 to ±40	−10.539	+1.5468	−0.0288	12420
±40 to ±90	−10.815	+1.6592	−0.0414	14730

With these data, computing the total number of stars for each of the three zones, I obtained

Gal. Lat.	No. Stars
0° to ±20°	43.8×10^9
±20 to ±40	3.6×10^9
±40 to ±90	0.043×10^9

whence

$$\frac{A}{D} = N = 47.4 \times 10^9. \qquad (6)$$

The remaining constants were so determined that (3) represents the values of G in Table III. I thus found

$$\begin{array}{ll} \text{Gal. Lat.} = 0^\circ & \text{Gal. Lat.} = 90^\circ \\ A = 0.110 & A = 0.376 \\ B = 1.30 \times 10^{-3} & B = 1.83 \times 10^{-3} \\ C = 0.657 \times 10^{-6} & C = 3.40 \times 10^{-6} \\ D = 2.32 \times 10^{-12} & D = 7.93 \times 10^{-12} \end{array} \Bigg\} \qquad (7)$$

The representation of the tabular values is as follows:

ρ	GAL. LAT. = 0°			ρ	GAL. LAT. = 90°		
	G. Tab.	G. Form.	Tab. − Form		G. Tab.	G. Form.	Tab. − Form
500......	30.0	30.2	−0.2	200......	54.3	50.1	+4.2
1000......	37.8	37.0	+0.8	500......	66.2	68.0	−1.8
2000......	36.0	35.0	+1.0	1000......	58.5	60.3	−1.8
3000......	31.7	30.2	+1.5	1500......	49.1	49.4	−0.3
4000......	26.0	26.0	0.0				
6000......	19.0	19.9	−0.9				
8000......	14.0	16.0	−2.0				

The agreement is not very good, but seems sufficient for our present purpose.

6. *Application of kinetic theory of gases.*—The results thus far obtained rest, it is true, on provisional data, which even now might be materially improved; they further depend on the supposition, not yet fully demonstrated, that, within the distances here considered, there is no appreciable extinction of light in space, but they are, nevertheless, I think, the legitimate outcome of our data.

For what follows I will now introduce some considerations borrowed from the kinetic theory of gases, the applicability of which to the stellar system might be considered doubtful. At all events I do not pretend to have demonstrated this applicability. The results which will be derived cannot lay claim to be demonstrably correct, but they seem to me to be so remarkable that, after a good deal of hesitation, I have resolved to publish them, in the hope that others, better versed in these matters, may furnish us with a more rigorous solution of the problem involved.

Even though it has been shown, in the main by unpublished investigations, that the peculiar motions[1] of the stars with some crude approximation are Maxwellian, the stellar system cannot be treated as a gas at rest; first, because of the existence of stream-motion; second, because of the form of the equidensity surfaces, which is certainly different from that of the equipotential surfaces of the gravitational force.

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[1] Peculiar velocity is defined as the motion corrected for both the solar and stream-motion. The radial and transverse velocities agree in showing a certain excess of very large motions over the Maxwellian distribution. They are both represented satisfactorily by the sum of *two* Maxwellian distributions. A thorough separate treatment of all the spectral classes is still a great desideratum.

That they are different is proved by the fact, among others, that in general the forces are not normal to these surfaces. This is evident enough without further explanation. Moreover, it is clearly brought out by Table III, where the angle with the normal reaches values of more than 27°. Further it is well known that in a gas at rest under its own attraction, the equidensity surfaces are spherical.

The system cannot therefore be in a steady state unless it has a systematic motion. Since the discovery of the star-streams it is clear that such a motion really exists and that it is parallel to the plane of the Milky Way.

It seems rational, therefore, to assume that the system has a sort of rotational motion round the *X*-axis (see Fig. 1) which is directed toward the pole of the galaxy. The form of the equidensity surfaces found directly in *Contribution* No. 188 as well as that now adopted. strongly indicates some such motion.

This being assumed, the stars along the axis will still have no other motion than their peculiar motions, which, as was just mentioned, are Maxwellian, at least with some approximation. I venture to assume, therefore, that the stars in the immediate neighborhood of this axis are arranged as the molecules of a gas in a quiescent atmosphere.

If:

Δ be the star-density (number of stars per cubic parsec);
u one of the components of the peculiar velocity;
η the acceleration produced by the attraction of a star of average mass at a distance of one parsec, then on the above assumption

$$\overline{u^2}\frac{\delta\Delta}{\Delta} = -G\eta\delta\rho, \tag{8}$$

$\overline{u^2}$ being the average value of u^2.

The formula is analogous to that used for barometric determinations of altitude in an atmosphere of constant temperature throughout. On the other hand, we have found empirically formulae such as (1) and (1*a*) (see also *Contribution* No. 188, p. 13 [21]); in other words,

$$\log \Delta = -P + Q \log \rho - R(\log \rho)^3 \quad (\rho > \rho_0), \tag{9}$$

from which, by differentiation

$$\frac{\delta\Delta}{\Delta} = \frac{Q - 2R \log \rho}{\rho}\,\delta\rho. \tag{10}$$

Comparing the two expressions, (8) and (10), for $\delta\Delta/\Delta$

$$G\eta = -\overline{u^2}\left[\frac{Q - 2R \log \rho}{\rho}\right] \quad (\rho > \rho_0). \tag{11}$$

As the motions are supposed to be Maxwellian, the well-known formula used in the theory of least squares gives

$$\overline{u^2} = \frac{\pi}{2}(\bar{u})^2. \tag{12}$$

From observations of the radial velocities at the Lick Observatory, where no choice has been made on the basis of proper motion (*Lick Observatory Bulletin*, **6**, 126), I derived the value[1]

$$\bar{u} = 10.3 \text{ km/sec.} \tag{13}$$

or since

$$\begin{aligned} 1 \text{ kilometer} &= 3.25 \times 10^{-14} \text{ parsecs} \\ 1 \text{ parsec} &= 3.08 \times 10^{13} \text{ kilometers} \end{aligned} \tag{14}$$

I find, in the units parsec and second, here adopted,

$$\bar{u} = 3.35 \times 10^{-13} \tag{15}$$

$$\overline{u^2} = 1.763 \times 10^{-25} \tag{16}$$

so that (11) becomes

$$G\eta = -1.763 \times 10^{-25}\,\frac{Q - 2R \log \rho}{\rho} \quad (\rho > \rho_0) \tag{17}$$

Finally, for galactic latitude 90°, we obtain from equation (1*a*) the values:

$$Q = +4.890 \quad R = +1.200 \tag{18}$$

$$\eta = \frac{(-8.620 + 4.229 \log \rho) \times 10^{-25}}{G\rho} \quad (\rho > 150). \tag{19}$$

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[1] There is a mistake in the derivation of this value. The true value is certainly somewhat lower. From considerations given below I have not deemed it necessary for the present paper to repeat the computations with an improved value of $\bar{u}$.

For small values of ρ, formula (9) does not hold. According to *Contribution* No. 188, and particularly according to *Contribution* No. 229, it represents the observations excellently for values of ρ well beyond the maximum (which in the present case lies near $\rho=110$ parsecs). For values of ρ below the maximum the density is nearly constant. The differential-quotient $\delta\Delta/\delta\rho$ thus becomes very small and $\delta\Delta/\Delta\delta\rho$ very unreliable. In the present case it will probably be well not to rely on the formula below, say, 150 parsecs. This limit was adopted in (19).

I have computed the values of η from (19) both on the supposition that G has the values found directly in Table III and that it has the values yielded by formula (3). The former were adopted (Table IV).

TABLE IV

VALUES OF η AND $\bar{m}$

Point	ρ	η Form. (3)	η Adopted	$\bar{m}$ (Sun = 1)
	parsecs			
II, 90°.........	198	11.1×10^{-30}	10.2×10^{-30}	2.2
IV, 90	413	$8.9 \quad 10^{-30}$	$9.0 \quad 10^{-30}$	2.0
VI, 90	717	$7.3 \quad 10^{-30}$	$7.5 \quad 10^{-30}$	1.7
VIII, 90	1114	$6.6 \quad 10^{-30}$	$6.8 \quad 10^{-30}$	1.5
X, 90	1660	$6.5 \quad 10^{-30}$	$6.5 \quad 10^{-30}$	1.4

The quantity η is, as stated above, the acceleration per second, in parsecs, produced by the attraction of a star of average mass on a body at a distance of one parsec. The acceleration which the sun would produce, expressed in the same units, is

$$\text{Acceleration by sun}=4.53\times10^{-30}. \tag{20}$$

This enables us to find the average mass $\bar{m}$ of a star expressed in the mass of the sun as a unit. The values of $\bar{m}$ thus found have been inserted in the last column of Table IV.

These values agree surprisingly well with what has been found by totally different considerations. In a recent paper[1] Jackson and Furner find for visual binary stars, as the best average

$$\frac{1}{\sqrt[3]{m_1+m_2}}=0.855.$$

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[1] *Monthly Notices*, 81, 4, 1920.

Consequently $m_1+m_2=1.60$, which agrees with Table IV if we suppose that the combined mass of the two components and not that of a single component is comparable with the mass of a single star, and especially if we further consider that there are theoretical grounds for expecting that the average mass will decrease for increasing distance.[1]

Remark. Dark matter. It is important to note that what has here been determined is the total mass within a definite volume, divided by the number of luminous stars. I will call this mass the average effective mass of the stars. It has been possible to include the luminous stars completely owing to the assumption that at present we know the luminosity-curve over so large a part of its course that further extrapolation seems allowable.

Now suppose that in a volume of space containing l luminous stars there be dark matter with an aggregate mass equal to Kl average luminous stars; then, evidently the effective mass equals $(l+K)\times$average mass of a luminous star.

We therefore have the means of estimating the mass of dark matter in the universe. As matters stand at present it appears at once that this mass cannot be excessive. If it were otherwise, the average mass as derived from binary stars would have been very much lower than what has been found for the effective mass.

7. *Angular velocities* (ω) *in the plane of the galaxy.*—Ignoring for an instant the fact that the stars in the Milky Way cannot be systematically at rest and treating the stars near this plane in the same way as those near the axis, I am led by a formula analogous to (17) to values of η which are not quite half those given in Table IV. I suppose that the difference must be wholly due to the centrifugal force induced by the rotational motions. In fact, I assume that the average mass is the same throughout the whole system, at least for points on the same equidensity surface.

If, therefore, ρ and ρ' represent the distances from the center, of two points on the same equidensity surface, the first in the direction of the Pole, the second in the Milky Way, for which points the total attractive forces of the system are respectively G and G', formula (11) gives

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[1] Jeans, *Problems of Cosmogony and Stellar Dynamics* (1919), p. 239.

$$\text{For the Pole} \qquad G\eta = -\overline{u^2}\,\frac{Q-2R\log\rho}{\rho}, \tag{21}$$

$$\text{For the Milky Way } G'\eta - \rho'\omega^2 = -\overline{u^2}\,\frac{Q'-2R'\log\rho'}{\rho'}, \tag{22}$$

where ω represents the angular velocity for the point in the Milky Way, and Q' and R' the constants of equation (9) for that same plane. The two equations determine η and ω. The equation for the latter is

$$\omega^2 = \frac{G'}{\rho'}\cdot\overline{u^2}\left[\frac{Q'-2R'\log\rho'}{G'\rho'} - \frac{Q-2R\log\rho}{G\rho}\right], \tag{23}$$

in which, for $\overline{u^2}$ see (16); for G' and G see Table V; and for Q and R see (18), and where finally, by comparing (9) and (1),

$$Q' = 2.368 \text{ and } R' = 0.593 \tag{24}$$

The maximum of Δ according to formula (9) lies, for the Milky Way at about $\rho = 100$, for the direction toward the Pole at $\rho = 110$. As has been mentioned, the formula ceases to be correct below these values. I assume, as before, that the limit of validity is $\rho = 150$.

8. *Angular velocity for stars not in the galaxy.*—For the regions not in galactic latitudes 0° or 90° I determine the angular velocity by the condition that the resultant of the attractive and centrifugal forces must be at right angles to the equidensity surfaces.

In order that the system may be in the steady state, I assume that the equidensity surfaces are at the same time equipotential surfaces for the resultant of the attractive and centrifugal forces. The above condition is implied in this assumption.

Since

X component of resultant acceleration $= -X\eta$,
Y component of resultant acceleration $= -Y\eta + \beta\omega^2$,

and since ψ is the angle between the normal and the X-axis, the equation for ω is

$$\beta\omega^2 - X\eta = -X\eta\tan\psi,$$

which becomes slightly more convenient by writing

$$X = G \cos \phi \qquad Y = G \sin \phi .$$

Whence

$$\omega^2 = \frac{G\eta}{\beta} \frac{\sin (\phi - \psi)}{\cos \psi} . \tag{25}$$

For the points in Table V, $\phi - \psi$ and ψ were taken from Table III. The computation was made with the aid of formulae (23) and (25).

TABLE V

VALUES OF ω^2

Point	ω^2	Point	ω^2	Point	ω^2
II, 0°.	0.1757×10^{-30}	II, 30°..	0.4066×10^{-30}	II, 57°.1..	0.4132×10^{-30}
IV, 0 .	$.0902 \quad 10^{-30}$	IV, 30 ..	$.2477 \quad 10^{-30}$	IV, 57.1..	.2835 10
VI, 0 .	$.03175 \quad 10^{-30}$	VI, 30 ..	$.1255 \quad 10^{-30}$	VI, 57.1..	.1606 10
VIII, 0 .	$.01237 \quad 10^{-30}$	VIII, 30 ..	$.0683 \quad 10^{-30}$	VIII, 57.1..	.0949 10
X, 0 .	$.00510 \quad 10^{-30}$	X, 30 ..	$.0376 \quad 10^{-30}$	X, 57.1..	.0568 10

For these same points I have furthermore computed the linear velocities in kilometers per second. They are in Table VI.

TABLE VI

LINEAR VELOCITIES

Point	β	$\beta\omega$	Point	a	β	$\beta\omega$	Point	a	β	$\beta\omega$
		km/sec.				km/sec.				km/sec.
II, 0°..	1010	13.0	II, 30°.	187	325	6.4	II, 57°.1	196	127	2.5
IV, 0 ..	2106	19.5	IV, 30.	391	677	10.4	IV, 57.1	410	265	4.4
VI, 0 ..	3657	20.1	VI, 30.	679	1176	12.8 (13.1)	VI, 57.1	711	460	5.7 (6.3)
VIII, 0 ..	5675	19.4	VIII, 30.	1055	1827	14.7 (15.7)	VIII, 57.1	1105	715	6.8 (9.1)
X, 0 ..	8465	18.6	X, 30.	1572	2733	16.3 (17.9)	X, 57.1	1647	1066	7.8 (11.1)

9. *Explanation of star-streaming.*—According to these numbers the angular velocity is not the same for the same distance β from the axis at different distances a from the plane of the Milky Way. Further on I will explain why the present solution must necessarily be a very crude one. For this reason I am not prepared to maintain the reality of this difference. On the contrary it seems very possible that a more definite solution will finally lead to the conclusion that all the points on a cylinder around the axis of the system move with the same velocity. In fact, if we base our solution on the equidensity surfaces as really derived from the observations,[1] instead

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[1] *Mt. Wilson Contr.*, No. 188; *Astrophysical Journal*, **52**, 23, 1920.

of assuming similar ellipsoids, we approach at once much closer to this state of affairs. The numbers in parentheses in Table VI give rough estimates of $\beta\omega$ based on this supposition. In the absence of a more definite solution I will confine myself mainly to points in the plane of the Milky Way, the motions of which, except for small values of β, seem to be somewhat better determined.

The most striking feature brought out by these numbers is undoubtedly the fact that at distances from the axis exceeding 2000 parsecs the linear velocity of the stars is nearly constant, the average being 19.5 km/sec.; that is, the great bulk of the stars must have a motion of 19.5 km in a direction parallel to the plane of the Milky Way. Observation has already proved that there really exists a systematic motion of the stars, that it is exactly parallel to the plane of the Milky Way, and that the motion takes place in two exactly opposite directions, the two streams having a relative velocity of about

$$40 \text{ km/per sec.} \tag{26}$$

Since in the preceding theory the motion is introduced simply to explain certain centrifugal forces, it is at once evident that it supposes nothing about the direction in which the motion takes place. Nothing prevents us from assuming that part of the stars circulate one way, while the rest move in the opposite direction. The relative motion of the two groups will then evidently be

$$2\times 19.5=39 \text{ km/sec.} \tag{27}$$

The motion to which our theory leads, besides being in the same plane, has therefore practically the exact value which is known from observation to exist. In fact we are led in the most direct and natural way to a complete explanation of the phenomenon of star-streaming. The circumstance that observation led us to assume two rectilinear streams, whereas we here find the motion to be circular, is probably unimportant. It is of course infinitely probable that the sun must be at a certain distance from the center of the system. If we suppose it to be at the point S (see Fig. 1) then the star-streams are derived from the observed motions of stars within a volume whose dimensions are of the order of those of the sphere around S shown in the figure. As long as the radius of this sphere is small in comparison with the distance of S from the center, the curvature of the stream-lines must be inappreciable.

When we consider that the value (27) has been obtained by a study of the arrangement of stars in space, in which the proper motions play no other part than that of a criterion of distance, while the value (26) has been obtained by a study of the motions themselves, both radial and transverse, the close agreement of the two results seems very significant. It becomes more so through the fact that both theories yield a motion exactly parallel to the plane of the Milky Way. Further, if we take into account the fact that the present theory leads to a value for the average mass of the stars which is in close accordance with what has also been found from utterly different investigations, and if we add a final point, namely, the natural explanation of the different arrangement of the stars of different spectral types, we are led irresistibly to the following conclusion:

The theory here propounded, though it may require considerable modification on account of its defectiveness both as to observational basis and mathematical treatment is probably correct in its main features.

The last point mentioned, which is open to quantitative verification, requires further investigation, but even now promises to be no less significant than the others. It is referred to again in section 14 below.

10. *Accelerations including centrifugal effect.*—If with the aid of Table V we now add the acceleration due to the centrifugal forces to that produced by the attractive force, the resultant will be in the direction of the normal. We find the results in Table VII.

TABLE VII

ACCELERATION, INCLUDING EFFECT OF CENTRIFUGAL FORCE

Point	$G\eta$	Point	$G\eta$	Point	$G\eta$	Point	$G\eta$	$G_{\text{Pole}}/G_{\text{MW}}$
II, 0°..	208×10^{-30}	II, 30°..	508×10^{-30}	II, 57°.1.	547×10^{-30}	II, 90°..	553×10^{-30}	2.7
IV, 0 ..	$131\quad10^{-30}$	IV, 30 ..	$495\quad10^{-30}$	IV, 57.1.	$577\quad10^{-30}$	IV, 90 ..	$593\quad10^{-30}$	4.5
VI, 0 ..	$90\quad10^{-30}$	VI, 30 ..	$367\quad10^{-30}$	VI, 57.1.	$458\quad10^{-30}$	VI, 90 ..	$480\quad10^{-30}$	5.3
VIII, 0 ..	$65\quad10^{-30}$	VIII, 30 ..	$268\quad10^{-30}$	VIII, 57.1.	$359\quad10^{-30}$	VIII, 90 ..	$384\quad10^{-30}$	5.9
X, 0 ..	$48\quad10^{-30}$	X, 30 ..	$193\quad10^{-30}$	X, 57.1.	$277\quad10^{-30}$	X, 90 ..	$302\quad10^{-30}$	6.3
ψ	90°		3°49′		1°25′		0°	

11*a*. *Velocity of escape.*—Assuming formula (3) to be valid for all distances, I find for the velocity of escape of a star near the center of the system

$$\left.\begin{array}{ll}\text{In the Milky Way} & 104 \text{ km/sec.}\\ \text{In the direction of Pole} & 98 \text{ km/sec.}\end{array}\right\} \qquad (28)$$

As a matter of convenience I used for this computation the constant value

$$\eta = 7.5\times 10^{-30} \qquad (29)$$

11*b*. *Velocity compared with circular velocity.*—The velocity of a star which moves in the plane of the Milky Way in a circular orbit is determined by the formula

$$V_c^2 = G\eta\beta. \qquad (30)$$

If for the point II, 0° we take $\eta = 10.2\times 10^{-30}$, and for the others the value (29), we obtain the results in Table VIII. The linear velocities $\beta\omega$ are thus seen to be well below the critical velocity V_c.

TABLE VIII

LINEAR AND CIRCULAR VELOCITIES IN PLANE OF MILKY WAY

Point	β	G	$\beta\omega$ Table VI	V_c
			km/sec.	km/sec.
II, 0°	1010	37.76	13.0	19.2
IV, 0	2106	35.71	19.5	23.1
VI, 0	3656	27.71	20.1	26.8
VIII, 0	5675	20.12	19.4	28.5
X, 0	8465	13.53	18.6	28.5

12. *Defects of solution.*—The way in which the values ω have been determined has made the resultant of gravitational and centrifugal forces perpendicular to the equidensity surfaces. In order that the surfaces may be really equipotential there is, however, a second condition to be satisfied, viz., that for points on the same surface the total force shall be inversely as the distance of two consecutive surfaces. This condition is *not* satisfied. The difference is shown, in its most extreme form, in the last column of Table VII, which, for the points in the Milky Way and in the direction of the Pole, situated on the same ellipsoid, shows the quotient

$$\frac{\text{Force at Pole}}{\text{Force in Milky Way (including centrifugal force)}}. \qquad (31)$$

If the equidensity surfaces were really accurately represented by concentric similar ellipsoids, similarly situated, as assumed thus far, this quotient in every case would be 5.1. In reality this is far from being so.

There are certain facts indicating that actually the equidensity surfaces cannot be similar ellipsoids as hitherto supposed in this article. It is a well-known fact[1] that the stars of large proper motion show no concentration toward the Milky Way, but are distributed over the various galactic latitudes with very approximate uniformity. The meaning of this can hardly be other than that the equidensity surfaces for the smaller distances are spherical. The equipotential surfaces of the gravitational and centrifugal forces, in order to coincide with the equidensity surfaces, therefore must also be spherical for vanishing distances; hence the quotient (31) must approach 1.00. Now this is just what the values of the quotient given in Table VII do. The change in these values is therefore an encouragement toward an attempt at an improved theory rather than otherwise.

13. *Position of the sun relative to the center of the system.*—Before such an attempt can be made with any hope of success it will be necessary, however, to free the data from several imperfections. Foremost among these is the imperfection in the adopted position of the sun. The data used in what precedes rest on the assumption that the sun is at the center. It seems infinitely improbable that this should be the case. Our theory, crude though it necessarily must be, paves the way for overcoming this difficulty:

First, as seen from the sun, the center of the system must lie in a plane at right angles to the true stream-motion. Adopting for the vertex of the relative motion of the two streams $\alpha = 6^h 17^m$, $\delta = +11°9$, or gal. long. = 167°, gal. lat. = 0°, we find that the center, as seen from the sun, must lie in gal. long. 77° or 257°.

Second, nearly all astronomers who have dealt with the question, though from a very different point of view, agree in assuming for the center a *southerly* galactic latitude.[2]

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[1] See for instance in *Verslag. Kon. Ak. v. Wetensch.*, Amsterdam, April, 1893, p. 137 (128).

[2] For instance, Struve, *Etudes d'Astronomie Stellaire*, pp. 61–62; Kapteyn, *Kon. Ak. Amsterd.*, April, 1893, p. 137; Hertzsprung, *Astronomische Nachrichten*, **196**. 207, 1914.

Third, Hertzsprung[1] from Cepheid variables finds the sun to be 38 parsecs north of the central plane of the Milky Way.

Fourth, the most difficult question probably will be that of the distance of the sun from the center. Still I think we can indicate a method which promises well.

For a first approximation, I start from the supposition, for want of a better one, that, even for the smaller distances, the *true* densities are as found in *Contribution* No. 188, which were derived on the basis of the erroneous assumption that the sun is the center. Further, for the moment, I will neglect the distance of the sun from the central plane of the Milky Way, which seems to be small.

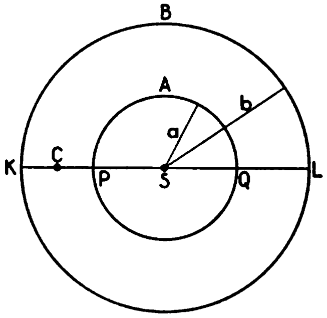

FIG. 2

In Figure 2, S represents the sun, C the center, and $CS=y$, the required distance. Let Δ_ρ represent the true star density at the distance ρ from the center. Now that we give up the erroneous supposition made in *Contribution* No. 188 and the present paper, we must consider the meaning of the densities given in Table VI of *Contribution* No. 188. What this table gives for a specified value of a (see Fig. 2) is in fact Δ', the average of the true densities at all points along the circumference PAQ around S with the radius a. This average will probably be not very different from the mean of the true densities at the points P and Q, and for the present, since our aim is only to arrive at a rough approximation, we will assume that it is exactly the case. Therefore, Δ being the true density,

$$\left.\begin{aligned}\Delta'_a&=\tfrac{1}{2}(\Delta_{y-a}+\Delta_{y+a})\text{ if }y<a\\ \Delta'_b&=\tfrac{1}{2}(\Delta_{b-y}+\Delta_{b+y})\text{ if }y>b\end{aligned}\right\}\qquad(32)$$

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[1] *Ibid.*, p. 208.

If now, as a first approximation, we take as the true densities those obtained by formula (1) and assume in succession $y=1500$, 1000, and 500 parsecs, we find values of Δ' corresponding to different values of a as given in Table IX.

Such a table then, or something like it, would have been obtained, instead of Table VI (lat. = 0°) of *Contribution* No. 188, if the center were actually at a distance y from the sun. Thus, if y were indeed 1500 parsecs, the considerations of *Contribution* No. 188 would have led to densities which, from distance *zero* up to distance somewhere near 1500 parsecs, would have increased. For $y=1000$ we should still have found an initial increase, and even for $y=500$ the same would have held. In this last case, however, the increase would have been so small that it might well have been overlooked owing to the errors of observation. Since in reality the observations lead to a regular decrease of the density throughout, I think we must conclude that y cannot have as high a value as 1500 or even 1000 parsecs. As an upper limit, we may take

$$y<700 \text{ parsecs.} \tag{33}$$

TABLE IX

VALUES OF Δ'

a (Parsecs)	y in Parsecs		
	1500	1000	500
0	0.25	0.40	0.73
250	.26	.45	.865
500	.29	.49	.70
1000	.43	.59	.49
1500	.55	.43	.29
2000	.30	.245	.19
3000	.145	.115	.10
4000	.079	.062	.055
5000	0.044	0.039	0.035

We can also find a *lower* limit from the condition that the relative velocity of the two star-streams must not deviate greatly from the value of 40 km/sec. derived from observation.

In the neighborhood of the sun the linear rotatory motion of the system, that is, one-half the relative velocity of the star-streams, is $y\omega$.

If this velocity exceeded the circular velocity of a star at distance y from the center moving under the attraction of the whole system, the system certainly would not be in a steady state. Hence, on the assumption of a steady state,

$$y\omega < V_c, \tag{34}$$

or by formula (30)

$$y\omega < \sqrt{G\eta y}\,. \tag{35}$$

Assuming that inside ellipsoid II η, has everywhere the constant value 10.2×10^{-30}, we thus find, expressing results in kilometers, for

$y=1000$ parsecs	$y\omega<19.1$ km/sec.
700 parsecs	<18.2 km/sec.
500 parsecs	<16.5 km/sec.

If, therefore, we admit that $2y\omega$, the relative stream-velocity, cannot well be below 35 km/sec., we obtain as a lower limit

$$y>600 \text{ parsecs} \tag{36}$$

The two limits (33) and (36) yield, as a first approximation,

$$y=650 \text{ parsecs} \tag{37}$$

All these results agree fairly well in locating the center, as seen from the sun, at

$$\left.\begin{array}{lll} \text{Gal. long. } 77^\circ & \text{or} & \alpha=23^h10^m \\ \text{Gal. lat. } -3^\circ & & \delta=+57^\circ \\ \multicolumn{3}{l}{\text{Distance projected on galactic plane } =650 \text{ parsecs}} \\ \multicolumn{3}{l}{\text{Distance projected at right angles to that plane } =38 \text{ parsecs}} \end{array}\right\} \quad (38)$$

The principal remaining uncertainty is perhaps that for the galactic longitude, which instead of 77° might be 257°. Personally I am strongly in favor of adopting the former value, which is in good accordance with the investigations of Herschel, Struve, and myself already quoted. My own result[1] was

$$\alpha=0^h, \quad \delta=+42^\circ.$$

The determination (38) lays claim to no accuracy. To improve it, it will be necessary to carry through a second, and perhaps a third and fourth, approximation, for the determination of the lower limit of y; besides, I think we may still improve or corroborate the values of the distance of the sun from the galactic plane.[1] Altogether, the location of the center would seem to be a laborious problem rather than one of great difficulty.

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[1] *Verslag. Kon. Akad. Amsterdam*, January, 1893, p. 129.

There is, moreover, the possibility of a direct determination. Given sufficient data on numbers of stars and proper motions for the regions around galactic latitude 0° and longitudes 77° and 257°, there would be no difficulty in deriving separately the star-densities at different distances from the sun in these regions. If there is truth in the above theory, these densities must *increase* with the distance up to distance y, in the direction *toward* the center, whereas in the opposite direction they must decrease with increasing distance. The two solutions together must yield a rather crucial test of the whole theory.

14. *Further defects. Separate treatment of the different spectral types.*—There are several further defects in the solution of the present paper. I will enumerate those that occur to me:

a) In the investigation on which the present paper is based the average parallax, as a function of magnitude and proper motion, has been taken from *Groningen Publication*, No. 8. At the present moment we have already available, though not yet published, much improved values, especially for the very small proper motions.

b) The value of $\overline{u^2}$ is not the very best that could have been obtained.

c) The values of G in Table III have been computed on the supposition that the average mass of a star is independent of the distance, whereas later, in section 8, it was found that this is not the case.

d) In this same computation for G, it was further assumed that the attraction of the whole of the system outside ellipsoid X on an internal point is zero. Though reasons were given for admitting that this attraction is small, it may not be negligible. Since according to what precedes it is highly improbable that the equidensity surfaces within X, as well as without, are really similar ellipsoids, the whole computation for G will have to be revised in a more definitive treatment.

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[1] Furthermore the distance of the center from the sun, as here found, is so small that, if special attention is given to the matter, it may not be hopeless to get evidence of the curvature of the stream-lines, particularly for the rather distant stars (faint stars having small proper motion).

e) The present solution is for all the stars together, that is, for a very heterogeneous collection of stars.

Only the last point is of fundamental importance, and I will devote a few lines to it. Just as in the higher, very attenuated parts of an atmosphere, where the molecular encounters become relatively rare and the different gases are, as it were, sorted out, so among the stars we must expect that the different spectral classes will show different arrangements in space, owing to the different values of $\overline{u^2}$ and η (mass) peculiar to these classes. The two cases are not identical, owing to the stream-motion in the stellar system, which is not supposed to exist in the atmosphere. Still there must be analogy. So, to take an extreme example, if there exists a class of stars for which $\overline{u^2}$ is zero, these stars, according to the present theory, would be confined exclusively to the central plane of the galaxy. For evidently a star not in that plane could not have pure stream-motion; that is, it could not move in a perfect circle around the axis and perpendicular to it, because the attractive force of the system on such a point does not lie in that plane. Therefore it would necessarily have a peculiar motion.

This simple consideration makes us understand at once the otherwise astonishing fact that the Wolf-Rayet stars lie with such close approximation in the central plane of the Milky Way. In the present theory this means simply that they can have no other than pure stream-motion (possibly with some peculiar motion *in* the plane of the Milky Way[1]). We thus realize that for the several classes of spectra there must be a very intimate connection between the values of $\overline{u^2}$ and η on the one hand, and the well-known differences in concentration toward the galaxy on the other.

Such considerations show that for the totality "all stars together" the value of $\overline{u^2}$ will doubtless change with the position in space and in particular with the distance of the stars from the plane of the Milky Way. The present investigation, therefore, suffers from the defect that this circumstance was neglected, $\overline{u^2}$ having been taken the same for the whole of the system. I think that this shows sufficiently the absolute necessity of treating the different spectral classes separately.

---

[1] It is noteworthy that, provided the peculiar motion *in* the Milky Way is also *zero* (as it must be when the Maxwellian distribution holds for this limiting case), the parallax of these stars becomes a pure function of the linear velocity, that is, of the quotient, radial velocity divided by sin λ, λ being the angular distance, star-apex.

For such a treatment the necessary data are not yet available, or at least not available in a satisfactory form. Fortunately, however, there exist at present instruments capable of dealing successfully with difficulties which, not so long ago, would have been insurmountable. With their aid we may hope to obtain material for several spectral classes separately, little or not at all inferior to what has already been obtained for the stars as a whole. I think that the determination of the spectral class for some 1000 stars of magnitudes 11 to 12, well distributed over the sky and for which the proper motion is either already sufficiently well known, or may be determined by taking a few additional photographic plates, will go far toward supplying us with what is so urgently wanted.

APPENDIX

Formulae for the computation of the attraction of a homogeneous revolution ellipsoid, of unit density (1 star per cubic parsec), on an exterior point.

Take the axis of revolution as X-axis and let the xy-plane contain the attracted point whose co-ordinates are a and β. Let A and B be the axes of the ellipsoid, the first coinciding with the axis of revolution, and let X and Y represent the components of the required attraction, Z being zero. Then from the well-known formulae,[1] if we put

$$\lambda^2 = \left(\frac{B}{A}\right)^2 - 1 \qquad (h)$$

$$K = \frac{2K}{\lambda^3}\left(\frac{B}{A}\right)^2 \qquad (k)$$

$$p = \lambda \frac{A}{A'}, \qquad (l)$$

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[1] I have used the formulae as given by Duhamel, *Cours de mécanique*, V, 1, pp. 321 and 318.

where, if $a=0$, A' is obtained from

$$A'=\sqrt{\beta^2-A^2\lambda^2} \tag{m}$$

and if $\beta=0$, from

$$A'=a. \tag{n}$$

For other values of a and β we may write

$$E^2=\frac{\lambda^2}{2}-\frac{a^2+\beta^2}{2A^2} \tag{o}$$

$$\eta=\lambda\frac{a}{A} \tag{p}$$

$$\left(\frac{A'}{A}\right)^2=-E+\sqrt{E^2+\eta^2} \tag{q}$$

With these auxiliary quantities we find

$$X=-aK\phi\ (p), \tag{r}$$

$$Y=-\beta K\omega\ (p), \tag{s}$$

in which $\phi(p)$ and $\omega(p)$, or rather their logarithms, have been tabulated in Table X by means of the formulae

$$\phi(p)=2[p-\text{arc tan } p] \tag{t}$$

$$\omega(p)=\text{arc tan } p-\frac{p}{1+p^2} \tag{u}$$

TABLE X

VALUES OF LOG ϕ (p) AND LOG ω (p) ACCORDING TO FORMULAE (t) AND (u)

p	Log ϕ	Log ω	p	Log ϕ	Log ω	p	Log ϕ	Log ω
0.000	$-\infty$	$-\infty$	0.05	5.920^{-10}	5.920^{-10}	0.5	8.861^{-10}	8.804^{-10}
1	0.824^{-10}	0.824^{-10}	6	6.157	6.156	0.6	9.076	8.997
2	1.727	1.727	7	6.358	6.357	0.7	9.252	9.149
3	2.255	2.255	8	6.531	6.530	0.8	9.399	9.272
4	2.630	2.630	9	6.685	6.682	0.9	9.524	9.372
5	2.921	2.921	10	6.821	6.819	1.0	9.633	9.455
6	3.158	3.158	11	6.945	6.942	1.1	9.728	9.525
7	3.359	3.359	12	7.058	7.054	1.2	9.812	9.585
8	3.533	3.533	13	7.161	7.157	1.3	9.886	9.635
9	3.687	3.687	14	7.257	7.252	1.4	9.954	9.679
10	3.824	3.824	15	7.346	7.341	1.5	0.015	9.717
11	3.948	3.948	16	7.430	7.423	1.6	0.070	9.750
12	4.061	4.061	17	7.508	7.500	1.7	0.121	9.780
13	4.166	4.166	18	7.581	7.573	1.8	0.168	9.806
14	4.262	4.262	19	7.651	7.642	1.9	0.211	9.828
15	4.352	4.352	20	7.717	7.707	2.0	0.252	9.850
16	4.436	4.436	21	7.779	7.768	2.1	0.289	9.869
17	4.515	4.515	22	7.839	7.827	2.2	0.325	9.885
18	4.590	4.590	23	7.896	7.882	2.3	0.358	9.900
19	4.660	4.660	24	7.950	7.935	2.4	0.389	9.914
20	4.727	4.727	25	8.002	7.986	2.5	0.418	9.927
21	4.790	4.790	26	8.052	8.035	2.6	0.446	9.939
22	4.851	4.851	27	8.100	8.081	2.7	0.472	9.950
23	4.909	4.909	28	8.146	8.126	2.8	0.497	9.960
24	4.964	4.964	29	8.190	8.169	2.9	0.521	9.969
25	5.018	5.017	30	8.233	8.210	3.0	0.544	9.977
26	5.069	5.068	31	8.274	8.250	3.1	0.566	9.985
27	5.118	5.118	32	8.314	8.288	3.2	0.587	9.993
28	5.165	5.165	33	8.352	8.325	3.3	0.607	0.000
29	5.211	5.211	34	8.390	8.361	3.4	0.626	0.006
30	5.255	5.255	35	8.426	8.396	3.5	0.645	0.012
31	5.298	5.297	36	8.461	8.429	3.6	0.663	0.018
32	5.339	5.339	37	8.495	8.461	3.7	0.680	0.023
33	5.379	5.379	38	8.528	8.493	3.8	0.697	0.028
34	5.418	5.418	39	8.560	8.523	3.9	0.713	0.033
35	5.456	5.455	40	8.591	8.552	4.0	0.728	0.038
36	5.492	5.492	41	8.621	8.581	4.1	0.743	0.042
37	5.528	5.528	42	8.651	8.609	4.2	0.758	0.046
38	5.563	5.563	43	8.680	8.635	4.3	0.772	0.050
39	5.597	5.596	44	8.707	8.662	4.4	0.786	0.054
40	5.630	5.629	45	8.735	8.687	4.5	0.799	0.057
41	5.662	5.661	46	8.761	8.712	4.6	0.812	0.060
42	5.693	5.693	47	8.787	8.736	4.7	0.825	0.064
43	5.724	5.723	48	8.813	8.759	4.8	0.837	0.067
44	5.754	5.753	49	8.837	8.782	4.9	0.849	0.070
45	5.783	5.782	0.50	8.861	8.804	5.0	0.861	0.072
46	5.812	5.811						
47	5.840	5.839						
48	5.867	5.866						
49	5.894	5.893						
0.050	5.920	5.920						

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A CONTINUOUS SPECTRUM FROM MERCURY VAPOR

By C. D. CHILD

ABSTRACT

Continuous spectrum of *mercury, red to ultra-violet.*—(1) *Conditions for emission in discharge tube.* A continuous spectrum identical with the fluorescence spectrum is emitted in addition to the ordinary line spectrum only when (*a*) the temperature of the vapor is above 120° C. and below 300° to 400°, depending on the pressure, when (*b*) the pressure is above 1 mm, when (*c*) the current-density is low, of the order of 10^{-4} amp/cm^2 or less and (*d*) when very little air or other contaminating gas is present. An influence machine or transformer is better as a source of current than an induction coil, especially one with condensers. Ionization of the vapor is not necessary for the emission of this spectrum since it was obtained with only 7 volts between the Wehnelt cathode and surrounding cylinder. (2) *In explanation* it is suggested that *the carriers of the spectrum* are molecules consisting of two or more atoms, which emit the fluorescence spectrum as a result of excitation by ultra-violet radiation λ 2536 emitted by atoms struck by electrons with energies corresponding to 4.9 volts or more.

Continuous spectrum of sodium.—Preliminary observations on heated sodium vapor in a discharge tube indicated that a continuous spectrum was present but much fainter than the line spectrum.

Ionization potential for mercury.—Some observations on the current from a heated filament to a concentric electrode as a function of voltage seem to show that the ionization potential *varies with the temperature*, reaching a minimum for about 140°, when it was 3 volts less than for 50°.

Change in chemical behavior of mercury vapor at about 120°C.—Above that temperature it *reacts with air* to form a dark compound, but not below. This reaction is associated with the appearance of the continuous spectrum.

The writer has recently observed that under certain conditions the electric discharge through mercury vapor produces a glow that is distinctly green. This shows the well-known line spectrum of mercury and also one that is apparently continuous from the red into the blue, being most prominent to the eye in the green. A brief mention was made by Warburg[1] in 1890, that he had observed a continuous spectrum when studying the electric discharge through mercury vapor. This was verified by Wiedemann and Schmidt,[2] Kalähne,[3] and Kowalski.[4] A continuous spectrum was also found by Hartley[5] when mercury vapor is excited to fluorescence by ultra-violet radiation, and was further studied by Wood[6] and others. However, no extended study has been made of this spectrum when excited by an electric discharge, and the investigation described in the following paragraphs was, therefore, undertaken.

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[1] *Wiedemann's Annalen*, **40**, 14, 1890.

[2] *Ibid.*, **57**, 457, 1896.

[3] *Ibid.*, **65**, 826, 1898.

[4] *Physikalische Zeitschrift*, **15**, 249, 1914.

[5] *Proceedings Royal Society*, A, **76**, 428, 1905.

[6] *Philosophical Magazine* (6), **18**, 240, 1909, and (6), **23**, 689, 1912.

Conditions necessary for obtaining the continuous spectrum. Small current-density.—There are three conditions necessary for obtaining the continuous spectrum by an electric discharge with any considerable brightness; the first being that the current-density must be small. As the current-density is increased the line spectrum becomes more prominent and the continuous spectrum less so. Contracting the tube as in the Plücker tube has the same effect as increasing the current.

The best source of current for showing this spectrum is the Wimshurst or Toepler-Holtz machine. With the current from such a machine it is often impossible to detect any lines even when the continuous spectrum is very bright. A step-up transformer may also be used to advantage. An induction coil is not so satisfactory. It is possible to obtain the continuous spectrum with the discharge from a hot wire covered with calcium oxide, if the current is sufficiently small. Putting condensers in parallel with the induction coil has the same effect as increasing the current. In most of the experiments described in this paper the discharge was obtained from a transformer, the current being approximately 5×10^{-5} amperes per square centimeter.

Temperature and density necessary for the continuous spectrum.—The second condition necessary for obtaining the continuous spectrum is that the temperature must be above 120° C. and below a higher temperature which depends on the density of the vapor. Kowalski found that mercury vapor excited by an electrodeless discharge gave a white glow at room temperature and a green glow when heated to 140° C. This is approximately the same temperature as that which was found necessary to produce the continuous spectrum.

A third condition necessary for the continuous spectrum is that the vapor have a density approximately that of saturated vapor at 120° C. This may be shown by maintaining the lower part of a tube containing a small amount of liquid mercury at a temperature below 120° C., while the upper part is kept above that temperature. The vapor in such a tube is less dense than that required and does not show the continuous spectrum. This will be discussed more fully in a following paragraph.

Even when the density of the vapor is kept constant the continuous spectrum becomes less prominent, if the temperature is raised too high. This was shown by a tube 25 cm long which was heated while being exhausted until all but a few minute drops of the liquid mercury were vaporized. It was then sealed off from the pump and further heated until the mercury was completely vaporized. This showed the continuous spectrum with scarcely a trace of the line spectrum. If it was further heated until the glass became slightly conducting the glow became distinctly blue with the discharge concentrated in a line in the center of the tube. The line spectrum was then prominent with but little of the continuous spectrum.

An experiment by Wood[1] showed that the same effect occurs at a higher temperature when the continuous spectrum is excited by ultra-violet light. In his experiment a quartz tube containing a very small drop of mercury was exhausted and sealed off from the pump and heated until the mercury was completely vaporized and the tube became red hot. It was found that after the mercury had all vaporized the fluorescence diminished in intensity and finally disappeared.

It is difficult to determine the exact temperature and density necessary to produce the continuous spectrum not only because any change in the spectrum due to heating the tube occurs gradually, but also because any trace of gas other than mercury hinders the production of the continuous spectrum. When a tube is sealed off from the pump and used for a time such gases are apt to occur. Lastly it is difficult to distinguish between effects due to temperature and those due to density, since it is impossible to increase the density of the vapor without increasing the temperature.

However, the arrangement shown in Figure 1 gives results of some value.

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[1] *Philosophical Magazine* (6), 18, 246, 1909.

The tube *T* was connected to the pump by a trap *d*. Observations were made with this open and with it closed. The upper parts of *T* and *c* were surrounded with an asbestos jacket which was heated by a ring burner. The lower part, including *d*, was immersed in an oil bath which was kept at a lower temperature. When the trap at *d* was open the tendency for the mercury to distil into the pump was hindered by placing a capillary tube 20 cm long and 1.25 mm in diameter at *c*.

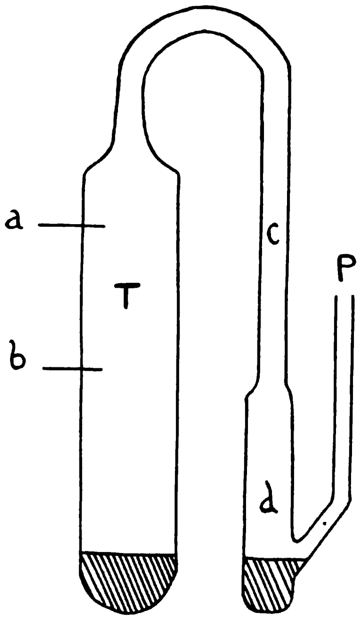

FIG. 1

The trap at *d* could be closed by distilling mercury over from the tube until the opening to *P* was closed. Both with the trap open and with it closed the tube was heated for some time before observations were taken.

It was found that both the upper and the lower parts of the tube must be at least at 130° C. in order to show the continuous spectrum prominently. Heating the upper alone was not sufficient even when the lower was but a few degrees below the required temperature. This again shows that it is not only necessary to have a required temperature but also a certain density of the vapor.

The lower part of the tube was then maintained at various temperatures above 130° C. and the upper part was heated until the continuous spectrum approximately disappeared. In this case the density of the vapor did not remain constant but diminished as the temperature of the upper part was raised. The results of such observations taken with the trap open are given in Table I but approximately the same results were obtained with the trap closed. Column 1 gives the temperature of the bath surrounding the bottom of the tube, and Column 2 gives the approximate temperature of the air inside the jacket above which the vapor ceases to give the continuous spectrum. Above 400° C. the glass became somewhat conducting, so that no higher temperature could be taken with the apparatus at hand.

TABLE I

Column 1	Column 2
132	310
140	335
150	375
160	410
167	440

As has been stated, the values given in the second column are rough approximations, but they are sufficiently accurate to prove that the temperature at which the continuous spectrum disappears depends on the density of the vapor in the tube. This is what might be expected if the continuous spectrum depends on the presence of molecules in the vapor. According to the mass law, if both atoms and molecules are present, the greater the density of the vapor the greater the relative number of molecules at a given temperature, and the higher the temperature needed to break up all of the molecules into atoms.

If both the vapor and the liquid mercury are heated above 200° C. the discharge becomes concentrated at the center of the tube. At 280° C. with a pressure of 15 cm the discharge is almost a line, which is very bright and shows the line spectrum much more prominently than the continuous spectrum.

Effect of vaporization and condensation.—It is easier to obtain the continuous spectrum where vaporization and condensation are taking place, as may be shown by heating a tube at some part where a few drops of mercury have condensed. If a discharge is passed through the tube, the continuous spectrum is much more prominent in the region where the vaporization occurs than elsewhere.

However, neither vaporization nor condensation is essential for the production of the continuous spectrum. This was shown in the experiment already described where a tube was heated until the mercury was completely vaporized. When there was neither vaporization nor condensation, there was still the continuous spectrum with scarcely a trace of the line spectrum.

There is, however, no question but that condensation aids in showing the continuous spectrum and also that it is the condensation rather than the vaporization that is helpful. It is somewhat difficult to prove this since newly formed vapor ordinarily expands as soon as it leaves the surface of the hot mercury and in doing so may condense enough to show the continuous spectrum. However, that the preceding statement is correct was shown by the tube illustrated in Figure 2. The electrode *a* extended into the liquid mercury. This was heated and the part of the tube at the left of *c* was kept as hot as possible. The part at the right was allowed to remain comparatively cool. The connection to the pump was left open, so that only a part of the vapor passed into the tube. Under these conditions the glow over the liquid mercury and as far as *c* was white, while that at the right of *c* was green. By having a current slightly greater than the minimum it is possible to obtain a discharge which shows no continuous spectrum at the left of *c* while it shows it very prominently at the right.

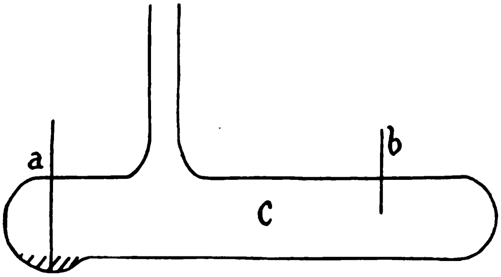

FIG. 2

Effect of other gases on the mercury vapor.—Wood[1] found that mercury would not fluoresce when mixed with other gases. The same has been found true when the continuous spectrum is produced by an electric discharge. As has been stated, a very slight amount of air changes very greatly the upper temperature at which the continuous spectrum disappeared. Again, with 2 or 3 mm of air and a small amount of mercury vapor it is impossible to obtain a continuous spectrum at any temperature. Thus in the preceding experiment, with the bath at the bottom of the tube at 140° C. and 3 mm of air present, no continuous spectrum could be obtained, whatever the temperature of the upper part of the tube. When the upper part was at 230° C. only the line λ 5461 could be detected. A somewhat similar phenomenon has been observed by Robertson[1] when exciting mercury vapor by electrodeless discharge.

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[1] *Astrophysical Journal*, 26, 43, 1907; *Philosophical Magazine* (6), 18, 244, 1909, and (6), 21, 314, 1911.

Continuous spectrum not affected by the kind of electrode, kind of glass, nor purity of mercury.—The continuous spectrum is not affected by the shape of the tube nor that of the electrode, providing the current-density is not increased. It is shown equally well with platinum, iron, or mercury electrodes and in tubes made of soda, lead, or Pyrex glass. It showed equally well with mercury that had been distilled three times and with mercury containing zinc, lead, cadmium, or sodium as impurities.

Mercury was also purified by distilling it in a stream of air. It is claimed by Hulett and Minchin[2] that mercury so distilled has less than one part of zinc to 10^{10} parts of mercury and that other oxidizable metals are removed to an equal extent. Mercury thus purified showed the same spectrum as other mercury.

Differences between normal mercury vapor and that giving the continuous spectrum. Ionization of the gas not necessary for radiation.—In a number of ways the vapor which is in a condition to give the continuous spectrum behaves differently from mercury vapor at room temperature. First it may be made to give visible radiation without being ionized. This might be expected from the fact that it fluoresces under the influence of ultra-violet radiation without becoming ionized, as has been shown by Wood.[3] That such radiation may also be produced by an electric discharge without ionizing the vapor is shown by the following experiment. Current from a hot wire covered with calcium oxide was passed to a surrounding cylinder. When the mercury at the bottom of the tube was heated so as to send a stream of hot vapor through the cylinder the region above the cylinder showed a faint glow which gave the continuous spectrum even when the voltage between the wire and the cylinder was much below that needed to produce ionization.

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[1] *Science*, **52**, 386, 1920.

[2] *Physical Review*, **21**, 388, 1905.

[3] *Philosophical Magazine* (6), **23**, 713, 1912.

In the experiment as performed the wire was 4 cm long and 2 mm in diameter. The surrounding cylinder was of iron, 5 cm long and 2 cm in diameter. At approximately 7 volts there was a faint glow above the cylinder, which grew somewhat brighter as the voltage was raised, but there was no large increase in the current until a difference of potential of 12 volts was applied, when the current suddenly increased and the region between the wire and the cylinder became luminous. The change above 12 volts was no doubt due to the beginning of ionization of the vapor. This would, therefore, seem to justify the conclusion that the continuous spectrum may be produced without ionizing the vapor.

A lower ionization potential.—Again the ionization potential of the vapor giving the continuous spectrum appears to be lower than that of normal mercury vapor. This was shown by examining at different temperatures the voltage at which there occurs a large increase in the current passing from a hot wire to a surrounding cylinder without any corresponding increase in the voltage. The same tube was used for this purpose as in the preceding experiment where the discharge from a hot wire was examined. The air was pumped out by a diffusion pump until no reading could be made on a McLeod gauge that read down to .0002 mm. The tube was then heated for some time so as to drive off any residual air, the tube becoming filled with mercury vapor from mercury kept in the bottom of the tube. The current through the platinum wire was kept at 4.5 amperes.

At approximately 50° C. the critical voltage was 16 volts. When the tube was heated until the continuous spectrum appeared the critical voltage was approximately 13 volts. On further heating until the mercury at the bottom was freely boiling the critical voltage increased to 18 volts. On allowing the tube to cool this voltage again decreased to 13 volts and then increased to 16 volts.

Data taken with different temperatures of the wire and with different sizes of wire all showed a minimum critical voltage when the tube was barely hot enough to show the continuous spectrum.

A check upon this is given by the fact that when a current is sent through a tube from a Wimshurst machine, the potential difference between the electrodes is smallest when the temperature is such that the continuous spectrum begins to appear. This was shown by rough measurements taken with an electroscope.

The high voltage required when the temperature was high was no doubt due to the large density of the mercury vapor. Although impact between electrons and atoms of mercury at low voltage is elastic, a small amount of energy is lost to the electron at each impact, so that with large density much of the energy is lost and a high voltage is necessary to produce ionization.

The decrease in voltage occurring when the temperature is such that the continuous spectrum is barely visible must, of course, be explained in some other way. It might be thought that at the lower temperatures there was some impurity in the gas which caused the critical voltage to be high. This does not, however, seem probable, first, because the amount of gas other than mercury vapor which was left in the tube was very small, its pressure being less than .0002 mm. Secondly, even this small amount would be swept out by the flow of mercury vapor into the cooler parts of the apparatus.

Again the lower critical value occurring when the continuous spectrum appeared might be explained on the ground that the platinum wire became hotter as the tube was heated and that the apparent critical voltage became lower because of the higher velocity of the emitted electrons. However, the resistance of the wire was measured by a Wheatstone bridge arrangement and it was found that when the tube was heated and the current through the wire was kept constant, the resistance of the wire became slightly less, indicating that the temperature of the wire decreased instead of increasing. This, no doubt, was due to the cooling effect of the mercury vapor about the wire. As the temperature of the tube is raised the density of the vapor increases and the denser vapor is able to conduct more heat from the wire, so that its temperature is lower, although the temperature of the surrounding tube is higher.

It, therefore, seems necessary to accept the view that the lower critical voltage is due to the fact that the ionization potential of the mercury vapor is less when the vapor is in a condition to give the continuous spectrum than when it is at room temperature. Whether this lowering is due directly to a change in the condition of the molecules which makes them more easily ionized, or to some ability which they acquire of absorbing radiation, cannot at present be stated. A more direct method of measuring the ionization potential is desirable in order to give us further knowledge of this subject.

Hot vapor active chemically.—During these experiments it was noticed that if any appreciable amount of air was left in the tube and the temperature raised sufficiently high to produce the continuous spectrum, a dark compound was formed which stuck to the sides of the glass. The substance was not formed when the discharge was passed through a tube at room temperature. By using a tube that was hotter at the bottom than at the top it was possible to observe the boundary between the region showing the continuous spectrum and that showing the line spectrum and it was found that this was also the boundary between the glass covered with the deposit and the clean glass.

Continuance of the glow.—When this discharge showed the continuous spectrum there was usually a green glow for several centimeters above the upper electrode. This could be explained as due to carriers of the continuous spectrum continuing to radiate for an appreciable time after they were excited. This continuance of the glow was examined by two different methods. One of those used has been described in a preceding article.[1] The discharge was passed from the secondary of a transformer and the time at which the current ceased was compared with the time at which the light ceased to be visible. This method indicated a continuance of the light for .002 seconds after the current ceased.

There is, however, an objection to this method owing to the possibility of the glass's becoming charged at the time the discharge passes and producing a luminous discharge after the current proper has ceased. The following arrangement obviates this difficulty to a large extent. The tube which has already been described in showing the discharge from a hot wire to a cylinder surrounding it was used. The current between the wire and the cylinder was made and broken by a wheel attached to the synchronous motor used in the preceding experiment. The difference in time between the breaking of the current and the time when the light ceased to be visible was apparently .008 seconds.

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[1] *Physical Review* (2), 9, 1, 1917.

Since the voltage here was not more than 30 volts and the wire was so nearly surrounded by the cylinder, it seems as if the glass could not have been charged to a sufficient amount to produce any light. On the other hand the fact that the light was here observed to continue for a period longer than that found with a spark discharge which is known to be somewhat untrustworthy makes it desirable to study this phenomenon further.

Character of the spectrum.—As far as could be determined by observations with the eye and by some rather crude photographs of the spectrum the continuous spectrum here studied is the same as that given by Wood[1] but the writer has neither the apparatus nor the experience to warrant further work with this. Wood's work has shown that this spectrum is continuous from the yellow into the ultra-violet at about the wave-length λ 3000 with a minimum at λ 3600. When the glow is produced by electric means and the conditions are at their best the red is also visible.

Again we might expect the relative intensity of different parts of the spectrum to vary as the conditions of the experiment varied. Such a variation was found by Wood[2] in the case of iodine vapor when its density was changed. A difference of this sort would indeed be expected if the molecules giving the spectrum differ in size or are subject to any change which progresses as the temperature is varied.

However, there does not appear to be any such variation in the continuous spectrum from mercury vapor, as is shown by the following experiments. A discharge was passed from the bottom of a tube containing boiling mercury to an electrode 24 cm above it. It is probable that if the molecules change either in size or in condition, they do so in passing from the hot liquid to the cooler region above. However, no difference could be observed in the character of the spectrum coming from different parts of the tube. Neither did changing the intensity of the light produce any change in the relative brightness of different parts of the spectrum.

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[1] *Philosophical Magazine* (6), **18**, 240, 1909. Plate 7, Fig. 6.

[2] *Ibid.* (6), **27**, 531, 1914.

Again the color of the discharge through tubes differing in temperature and in density of vapor is the same as long as only the continuous spectrum appears. For example one tube was heated to approximately 140° C. and second to 220° C. The pressure of the vapor in the first case was 2 mm and in the second was 3 cm. When the two tubes were placed side by side and the same current sent through them no difference in color could be detected. At higher temperatures the discharge becomes concentrated in the center of the tube and the line spectrum becomes noticeable. In that case a difference of color is, of course, noticeable.

Moreover, Wood and others who have examined the fluorescence of mercury vapor when at atmospheric pressure mention the color as being green, which is the same as that given by the vapor at low pressure.

The evidence obtained from the spectrum would, therefore, indicate that the oscillating system which gives the continuous spectrum is distinctly different from that which gives the line spectrum, that the light always comes from the same oscillating system rather than from a number of different ones and that it is not affected by the frequency of the impacts between the molecules.

The carriers of the continuous spectrum.—In attempting to determine the carriers of the continuous spectrum two explanations have been considered. One is that the same vibrating system gives both the line and the continuous spectrum, the vibrations being modified when the vapor is dense by the greater number of impacts. There are two objections to this view. First, experiment shows that when the density is kept constant and the temperature of the vapor is raised the ability of the vapor to give the continuous spectrum is impaired, if not destroyed. But under these conditions there is an increase in the number and intensity of impacts per second and if the continuous spectrum is due to impacts between atoms, one would expect the continuous spectrum to become more conspicuous instead of less so.

Secondly, there are impacts at 120° C. as well as at 140° C. They are indeed more than twice as frequent in the latter case and somewhat more intense, but if the impacts affect the oscillations at one temperature, one would expect some effect at the other and yet at the lower temperature there is usually no indication of the continuous spectrum, while at the higher one it almost entirely displaces the line spectrum. Moreover, the usual change in the spectrum to be produced by increasing the density is a broadening of the lines and not a complete change of the spectrum.

It seems to the writer that a more probable explanation is that the carriers of the continuous spectrum are molecules of mercury while the atoms are the carriers of the line spectrum. The determination of the specific gravity of mercury vapor indicates that it is monatomic, but these determinations are not sufficiently accurate to make the presence of the molecules seem improbable even when the temperature is high, and it is difficult to believe that a monatomic vapor can give two such radically different spectra and behave in other ways as differently as this vapor does.

Such an assumption would explain the fact that with high temperatures or with large currents such as occur with the mercury arc the continuous spectrum does not appear, for in such cases the molecules are no doubt split up into atoms which give only the line spectrum.

This, however, does not explain the large change in the spectrum which occurs at 130° C. We are, therefore, compelled to make the further assumption that the molecules are more easily excited at high temperatures than at low. Such an assumption is not unreasonable. While energy is probably not absorbed by the internal structure of isolated atoms when the gas is heated, it probably is absorbed by molecules of two or more atoms. If so, such molecules may be more easily ionized and excited to radiate at higher temperatures. According to such an assumption, at room temperatures the atoms are the more easily excited and only the line spectrum appears, while at the higher temperatures the molecules are more easily excited and we have also the continuous spectrum. The facts can, therefore, be explained on the following assumptions: *First, the carrier of the continuous spectrum are molecules consisting of two or more atoms, and secondly, these molecules are more easily excited at high temperatures than at low.*

Manner of excitation.—There are at least two ways in which the vibrations giving the continuous spectrum may be excited; the molecules may be agitated by impacts of electrons on the molecules, or by the absorption of ultra-violet radiation. The impact of electrons on the atoms of mercury no doubt produces radiation of wave-length λ 2536 and in a similar way the impact on molecules may produce the continuous spectrum, but since this spectrum is certainly produced by ultra-violet radiation, as was shown by Wood, and since radiation of wave-length λ 2536 is produced by the impact of electrons on atoms as has been shown by Franck and Hertz, it seems to the writer more probable that there is first a production of the radiation of wave-length λ 2536 and that this under proper conditions causes fluorescence of the vapor which gives out the continuous spectrum.

When an electron which has passed through a difference of potential of 4.9 volts, hits a mercury atom, it gives up its energy to the atom and the atom radiates this energy as ultra-violet light. When the gas is as dense as in the cases here studied the greater part of the energy goes primarily into such radiation, for with such densities the electrons will in almost every case hit an atom after it has passed through 4.9 volts and before it has passed through 10.4 volts, which is the voltage necessary to produce ionization.

An atom which has thus been partially ionized may be hit by a second electron before it radiates its energy and complete ionization may occur, as has been pointed out by Van der Biji[1] and Compton.[2] The recombination resulting from such ionization produces the line spectrum of mercury. With larger currents the chance of an atom being hit a second time before it has radiated its energy is, of course, greater than with small currents, so that entirely aside from the increase in temperature caused by the large current the line spectrum is more apt to be prominent when the current is large.

On the other hand, if the partially ionized atom is not quickly hit, it will radiate its energy. This may pass to a second atom and partially ionize it, which in turn may give out this same kind of radiation. This last is the kind of fluorescence which is known as resonance radiation.[1]

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[1] *Physical Review* (2), 10, 546, 1917.

[2] *Ibid.* (2), 15, 476, 1920.

If there are no molecules present, or if they are not in a condition to absorb this energy, as they apparently are not at room temperature, then all of the energy of this radiation will either aid in ionizing atoms or will be radiated from the gas as wave-length λ 2536. If, however, there are molecules of two or more atoms present which are in a condition to absorb the energy of this radiation, they fluoresce, emitting a radiation which gives the continuous spectrum.

The manner of excitation of the continuous spectrum is then conceived to be as follows: The electrons partially ionize some of the atoms. These radiate energy of wave-length λ 2536. If this radiation falls on molecules which are capable of absorbing it, the vapor fluoresces, giving out the radiation which gives the continuous spectrum.

The experiment which has been described showing that it is possible to produce the continuous spectrum without ionization confirms this explanation.

The fact that the line spectrum appears at the cathode even when the continuous spectrum appears in the positive column may be explained by assuming that in the cathode drop where the electric force is large the electrons attain sufficient velocity to completely ionize the atoms upon impact, and the resulting recombination of the positive ions with the electrons gives the line spectrum.

The condensed discharge.—As has been stated the discharge from an induction coil does not give as prominent a continuous spectrum as that from a Wimshurst machine. This is especially true, if condensers are in parallel with the tube. A tube so heated that the discharge from a Wimshurst machine will show only the continuous spectrum will scarcely show anything but the line spectrum with the spark from an induction coil.

There are two respects in which the condensed discharge differs from the uncondensed. The maximum current is greater with the condensed discharge, and the voltage when the spark commences is higher due to the fact that the ions have nearly all recombined since the preceding discharge. The following experiment indicates that it is the large current rather than the high voltage which causes the difference. The current from a Wimshurst machine was passed between *a* and *b* in Figure 3 when the tube was so heated as to give the continuous spectrum. A spark from an induction coil having capacity in parallel with the tube was then passed between *c* and *d*. This was found to show the line spectrum as plainly between *a* and *b* as in the other parts of the tube. The voltage between *a* and *b* could not in this case have been greater with the spark since the glow discharge would keep the gas in this region conducting while the current from the induction coil was interrupted.

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[1] *Philosophical Magazine* (6), 23, 699, 1912.

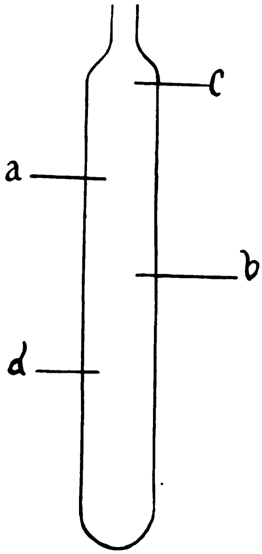

FIG. 3

Continuous spectrum from other gases.—If this explanation is correct, it would be reasonable to expect a continuous spectrum from other gases, or at least with other monatomic gases when near the point of condensation. Some observations were made on the discharge through sodium vapor when heated. There appeared to be a continuous spectrum present but it was much fainter than the line spectrum, so that no very definite description of it can be given. It is probable that it was the fluorescent spectrum of sodium vapor which has been described by Wiedemann and Schmidt[1] and others.

I desire to express my appreciation of the assistance which has been given me in the preceding work by Mr. B. K. Northrup, of Cornell University, and Professor A. E. Wood, of Colgate University.

COLGATE UNIVERSITY
September 1921

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[1] *Annalen d. Physik*, **56**, 18, 1895.

FORMATION AND LIFE OF METASTABLE HELIUM

By FABIAN M. KANNENSTINE

ABSTRACT

Current-potential curves for alternating and intermittent arcs in pure helium between a Wehnelt cathode and nickel anode were obtained by use of a Braun-tube oscillograph, for pressures of from 0.06 to 2 mm. With alternating voltages, the striking potential for each half-cycle was 29 volts or more for frequencies of 200 or less, but for frequencies of 220 cycles and above, the current started at about 5 volts and soon reached a saturation value until about 25 volts caused a further increase. In all cases the breaking potential was about 5 volts. To form the intermittent arc, the voltage was periodically dropped from 36 to some lower predetermined value. If this lower potential was less than 4.6 volts, the arc was suddenly extinguished; if between 4.6 and 23.7 volts, the arc died out gradually; and if it was 25.3 volts or above, the arc was maintained. All the foregoing values include a correction of 1.1 volts added to the actual readings to bring them in agreement with accepted results.

Evidence as to the life of metastable helium.—To explain various experimental results, it has been suggested that atoms of helium, partially ionized by electronic collisions of 20.8 volts or more and hence capable of complete ionization by further impacts of 4.8 volts or more, remain in this metastable condition for an appreciable time. The foregoing results not only confirm this suggestion but also show that under the circumstances of this experiment the metastable form persists for about 0.0024 second.

INTRODUCTION

Experiments with electron impacts in helium have led to a value for the minimum ionizing potential of 25.3 volts and to two radiating potentials, one at 20.5 volts[1] and another at 21.3 volts.[2] Franck and Reiche[3] and Franck and Knipping[2] conclude that the partially ionized state formed when the atom has been excited by the impact of 20.5 volt electrons is a "metastable" state, which may persist for some time and may be even permanently stable in absolutely pure helium. This conclusion is drawn from the fact that the radiation usually observed with electron impacts at this voltage disappeared when the helium was very pure, indicating that the displaced electrons do not return to the normal state. In slightly impure helium they also observed that the radiation, as measured by its photo-electric effect, appeared to continue for a brief time after the exciting electron impacts ceased.

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[1] F. Horton and A. C. Davies, *Philosophical Magazine*, **42**, 746, 1921; J. Franck and P. Knipping, *Physikalische Zeitschrift*, **20**, 481, 1919; K. T. Compton, *Philosophical Magazine*, **40**, 553, 1920; F. S. Goucher, *Proceedings Physical Society of London*, **33**, 13, 1920.

[2] J. Franck and P. Knipping, *loc. cit. Zeitschrift für Physik*, **1**, 320, 1920. F. Horton and A. C. Davies, *loc. cit.*

[3] *Zeitschrift für Physik*, **1**, 54, 1920.

Experiments by Paschen,[1] which show that in pure excited helium resonance radiation may be caused by the infra-red line 10830 A, also support the conception of a metastable state. All the energy of this wave-length absorbed by the excited atoms was found to be re-emitted as radiation of this same wave-length. In terms of Bohr's model this means that the radiation is absorbed by atoms which have already absorbed the energy corresponding to the radiating potential 20.5 volts, and that the displaced electrons return to this partially ionized state only, and not to the normal state.

The semi-stable existence of partially ionized helium atoms is also useful in explaining the phenomena in low-voltage arcs in helium, especially the striking of the arc at 20.5 volts with dense electron currents.[2] The phenomena of successive impacts as discussed by Compton[3] are of much more importance when the partially ionized atoms have a relatively long life. The metastable helium should be completely ionized by electrons with a speed corresponding to a potential of $25.3-20.5=4.8$ volts. If we assume that the metastable state may be produced by the arc radiations,[4] then an arc might be maintained with dense enough electron currents down to 4.8 volts. It is of interest that in this way Compton, Olmstead, and Lillie have maintained an arc with potentials as low as 8 volts between the electrodes. Horton and Davies[5] report an unsuccessful attempt to observe the ionization of abnormal helium atoms by bombardment with electrons of 4.8 volts.

In this paper experiments with helium arcs are described which show that helium may exist temporarily in a metastable form, and that arcs may be maintained in it with 3.5 volts. An approximate determination of its life is also reported.

[1] *Annalen der Physik*, **45**, 625, 1914.

[2] K. T. Compton, E. G. Lillie, P. S. Olmstead, *Physical Review*, **16**, 282, 1920.

[3] *Physical Review*, **15**, 476, 1920.

[4] O. W. Richardson and C. B. Bazzoni, *Nature* **98**, 5, 1916; F. Horton and A. C. Davies, *op. cit.*, **42**, 752, 1921.

[5] *Ibid.*, **42**, 763, 1921.

ARCS WITH LOW-FREQUENCY VOLTAGES

The arcs were formed between a Wehnelt cathode and nickel anode in a pyrex tube containing carefully purified helium. The helium was purified by passing over heated copper oxide and charcoal immersed in liquid air. Before passing into the experimental tube the helium passed through two additional charcoal tubes and a liquid-air trap. The tube and liquid-air trap were baked out at 450° C. for several days before the helium was admitted.

In the first experiments the helium was excited by an alternating e.m.f. An electromagnetic oscillograph was used to obtain curves showing the variation of current and voltage with time. From such curves the current-voltage curve may be obtained.

A decided dissymmetry in the growth and decay of the arc current could be observed, and also an apparent continuation of the arc current down to very low voltages. To obtain greater sensitiveness and to avoid the possible distortion from the inertia of the oscillograph elements, a sensitive Braun tube oscillograph was used in most of the later experiments. The electrostatic deflecting plates of the Braun tube were connected across the anode and cathode of the helium tube, while the arc current passed through the electromagnetic deflecting coils. The electrostatic and electromagnetic deflections were at right angles, so that the figure on the screen of the Braun tube was a graph in rectangular co-ordinates of the impressed e.m.f. and the arc current. The deflecting coils were so adjusted that the thermionic current alone did not produce any observable deflection.

Figure 1 shows the kind of curve obtained with a 60-cycle e.m.f. in helium at a pressure of 2 mm. It is seen that at a certain voltage the arc strikes, and as the voltage increases the current increases also. When the voltage decreases the current decreases at the same time, but continues with considerable magnitude until the voltage reaches a value of about 4 volts, when the current ceases to flow. Careful observation showed a sudden drop to zero at this lower voltage. This continuation of the arc to very low potentials may be attributed to the persistence of a metastable form of helium in which an arc may be maintained down to about 4 volts. Two other breaks in the down curve may be observed just below the striking potential of the helium itself. These breaks will be referred to later.

This type of curve was observed with 60 cycles at pressures down to 0.06 mm. Down to 1 mm pressure the striking potential was about 28 volts, and at the dying-out potential of about 4 volts a very decided drop of the arc current to zero was observed. As the pressure was decreased below 1 mm the striking potential increased and the dying-out potential became less certain, i.e., the drop to zero current was not so sudden as at higher pressures. At 0.06 mm the striking potential had increased to about 40 volts and the dying-out potential to about 20 volts. Figure 2 shows the type of curve obtained at 0.06 mm pressure. At all these pressures the filament temperature was approximately constant. The mean arc current varied from 25 milliamperes at 2 mm pressure to 10 milliamperes at 0.06 mm pressure.

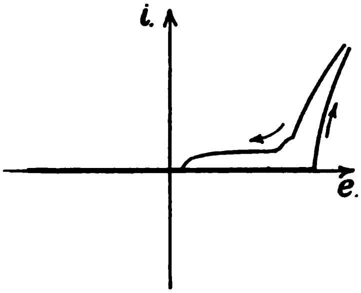


FIG. 1

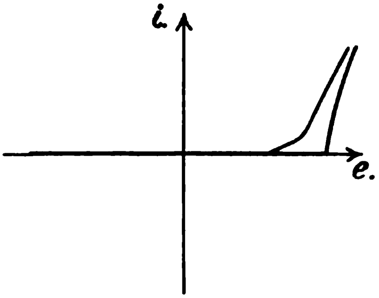


FIG. 2

If any impurities are present one would expect their effect to be more marked at lower pressures. Franck and Knipping have assumed that metastable helium will not return to normal helium except in the presence of impurities, so that with larger amounts of impurities we might expect the metastable helium to be destroyed in a shorter time. This is probably the interpretation of the gradual dying out of the arc before the voltage had dropped to 4 volts, as was observed when the pressure was very low. Experiments were made directly with the method discussed in the next section to see if the life was affected by impurities obtained by removing the liquid air from the charcoal for a short time till the helium had become very impure. The experiments indicated that while the impurities suppressed the amount of the metastable helium, the part remaining had the same life as that formed in the pure helium. The increase in the striking potential as the pressure was reduced is more difficult to interpret and is probably connected with the mechanism of the arc formation.

ARCS WITH HIGH-FREQUENCY VOLTAGES

With a sufficiently high frequency the e.m.f. should pass from +4 volts to its negative maximum and back to +4 volts before the metastable helium has disappeared, and upon reaching 4 volts the arc should strike, due to the ionization of the metastable helium still present.

Accordingly various frequencies from 60 to 10,000 cycles were impressed on the arc. An audion oscillator was used as a source for these frequencies. The author wishes to acknowledge his indebtedness to Dr. J. P. Minton and Mr. V. O. Knudsen for the use of their oscillators in this work. The alternating e.m.f. impressed on the arc was obtained from the secondary of an iron core transformer having its primary winding in the plate circuit of the oscillator.

Figure 3 shows the type of figure observed above 220 cycles per second at 2 mm pressure. The arc struck at about 4 volts, and the current rapidly attained a saturation value at which the arc current continued until the voltage had reached about 25 volts, when a second sudden increase was obtained. The decreasing part of the curve was above the increasing part but fell to zero at the lower striking voltage.

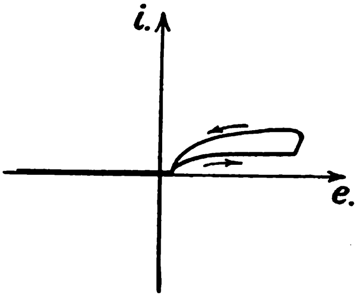


FIG. 3

With 200 cycles no appreciable arc formation at the lower voltage could be observed, indicating that the metastable helium practically disappeared in less than approximately $\frac{1}{400}$ of a second. The striking of the arc at about 4 volts with 220 cycles indicates that enough of the metastable helium was left after approximately $\frac{1}{440}$ of a second to give an observable indication of ionization.

All the figures above 220 cycles were similar, except that as the frequency was increased the striking at about 4 volts became more and more marked.

In using the oscillator as a source of alternating e.m.f. smooth curves were not always obtained. As the mean arc current was increased, by raising the electron emission, certain parts of the curve became wavy. This was particularly noticeable on the striking of the arc at about 25 volts. Instead of the arc current increasing as in Figure 1, it increased in a wavy form, indicating that the impressed voltage rose and fell during this striking period. This is provisionally attributed to self-induction in parts of the circuit or to the presence of iron in the oscillating circuit.

ARCS WITH INTERMITTENT CONSTANT VOLTAGES

In another series of experiments intermittent constant voltages from a battery were impressed between the hot cathode, *C*, and the anode, *A* (see Fig. 4). When the mercury interrupter, *H*, was open the voltage impressed on the electrodes was about 35 volts, and when the interrupter shorted the section *BD*, the voltage dropped to a value dependent upon the position of the slider *B*. In this way it was possible to impress a voltage of about 35 volts and suddenly change this voltage to a lower value. The arc current was passed through the deflecting coils, *M*, of the Braun tube. The spot was examined directly and also with a rotating mirror. Since it was rather difficult to observe the spot in the rotating mirror the following arrangement was also used. A uniform time deflection was obtained by connecting the deflecting plates to a potentiometer with a rotating contact, which was mounted on the shaft of the mercury interrupter. The electrostatic deflection was therefore proportional to the time and the

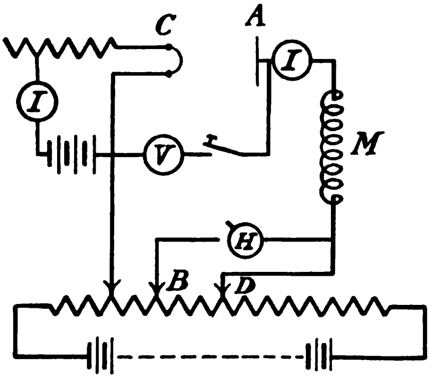


FIG. 4

magnetic deflection proportional to the current through the arc. These two deflections were at right angles. Furthermore, since the rotating contact was mounted on the interrupter shaft, the figure produced was stationary and could be carefully examined as the slider *B* was moved.

With the lower voltage below 3.5 volts and without the time deflection two spots appeared on the Braun tube screen, the upper one corresponding to the current for 35-volt excitation and the lower one for zero current. Using the time deflection, it was seen that the arc was extinguished in a time too short to be so observed, as shown at the right of Figure 5. Only a very faint vertical line appeared, since the spot moved very rapidly from the upper to the lower positions.

As the lower voltage was increased to 3.5 volts the lower line suddenly became curved as shown in Figure 6, indicating a gradual decrease of the current to zero value. This sudden curving of the lower line at 3.5 volts may be attributed to the presence of metastable helium produced during the 35-volt excitation. That is, the voltage dropped from 35 volts to 3.5 volts in a time short compared with the life of the metastable helium, and the arc was maintained by 3.5 volts as long as the metastable helium was present.

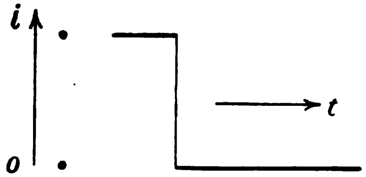


FIG. 5

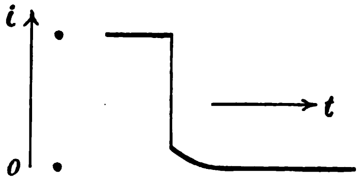


FIG. 6

The dying out of the arc continued to have the same appearance until the lower voltage had been increased to 22.6 volts, when a third spot appeared slightly above that for zero current as in Figure 7. Using the time deflection, it was seen that the arc was maintained for a brief time with this voltage before dying out.

A slight further increase of the lower voltage to 23.6 volts brought out still another spot, as in Figure 8, and the time deflection showed a dying out of the current in two stages as shown. The voltages for these two breaks in the decrease correspond to the voltages for the two breaks observed in the down curve of the alternating e.m.f. figure already described. This decrease in two stages may be connected with the fact of the existence of two radiating potentials of helium 0.8 volts apart corresponding to the first coplanor orbit (helium) and the first crossed orbit (parhelium) on Franck and Knipping's interpretation.

With a slightly increased lower voltage of 24.2 volts the arc was maintained. The two lower spots of Figure 8 disappeared, and the arc current jumped between the two values indicated by the two upper spots.

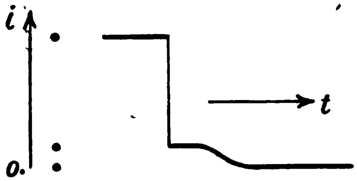


FIG. 7

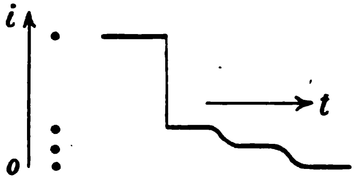
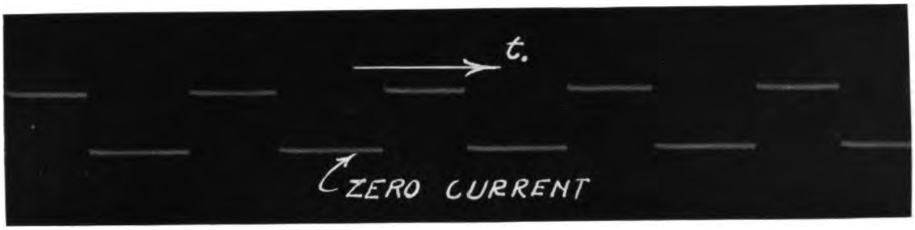


FIG. 8

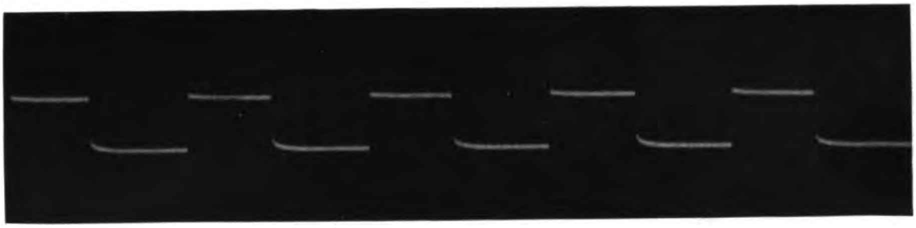
These phenomena were qualitatively the same at pressures ranging from 2 mm down to 0.06 mm. The above exact determinations were all made visually with the Braun tube oscillograph, but for the purpose of illustration photographs made with the electromagnetic oscillograph show qualitatively the same phenomena, although with less sharpness in the details. Plate VI shows the various changes which occur as the lower voltage is increased from zero to 24.2 volts. These oscillograms were obtained by passing the arc current through an element of the electromagnetic oscillograph and photographing the curve while the mercury interrupter was operating. In *a* the lower voltage was below 3.5 volts, and it is seen that the arc current drops to zero in a very short time. The curving of the lower current line when the lower voltage reaches 3.5 volts is shown in *b*. The brief maintenance of the arc at a lower voltage of 22.6 volts is shown in *c*, and the decrease in two stages at a lower voltage of 23.6 is shown in *d*. The maintenance of the arc at 24.2 volts is shown in *e*.

A number of readings were taken of these voltages, and then the cathode current was reversed in order to eliminate the drop in voltage along the cathode, and another set of readings taken. The

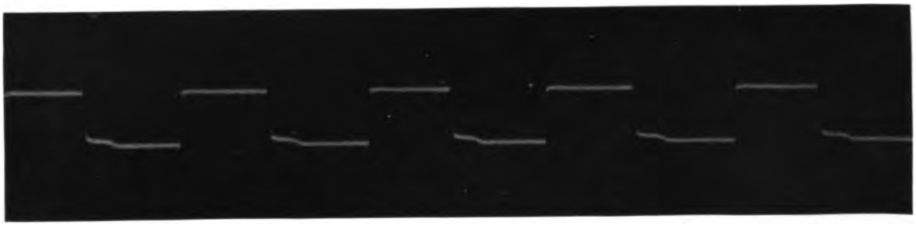
PLATE VI



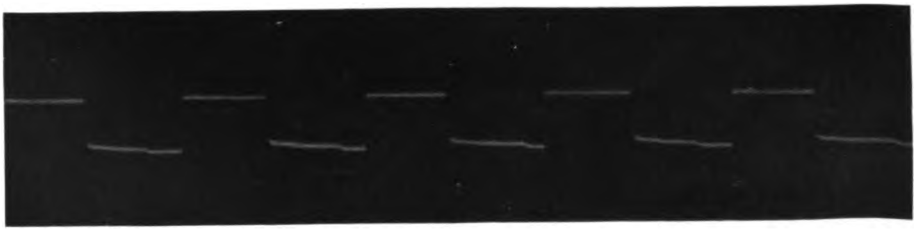
a



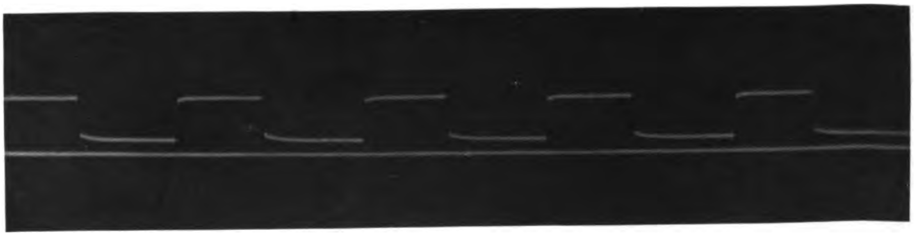
b



c



d



e

above values are the mean of all these values and thus are the potentials correct to approximately 0.1 volt between the anode and the center of the cathode. Corrections are necessary to allow for the initial velocity of emission of the electrons, the contact difference of potential, and the fact that some of the electrons are emitted from a more negative part of the filament than the center. As the drop of potential from the center to one end was 0.9 volt, this last correction may amount to about 0.4 volt. The other two corrections may easily amount to one volt or more according to a recent theory of K. T. Compton[1] in explanation of the results obtained by Hebb[2] on the arcing voltage in mercury vapor. It is thus permissible to apply a correction of 1.1 volts to the observed values given to make the upper voltage required to maintain the arc, 24.2, agree with the observed value of the ionization potential in helium, viz., 25.3 volts. This correction applied to the lowest potential observed for striking or maintaining the arc, 3.5 volts, gives a value of 4.6 volts in agreement within experimental error, with the theoretical ionization potential for metastable helium. Or, avoiding the question of corrections entirely, we may say that the difference between the two voltages observed, the lowest potential required to maintain an arc in helium with low electron density, and the minimum potential to maintain an arc in metastable helium is $24.2-3.5=20.7$ volts, and this value is within experimental error the minimum potential required to produce the metastable form of helium.

The author wishes to express his thanks to Professor H. G. Gale for his kind encouragement and advice during the progress of this work, and to Professor A. J. Dempster, who proposed the investigation of this particular phase of helium arcs, and without whose suggestions and assistance this work would not have been possible.

RYERSON PHYSICAL LABORATORY
April 1922

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[1] American Physical Society Meeting, Washington, April 22, 1922.

[2] *Physical Review*, **16**, 375, 1920.

NOTES ON IONIZATION IN THE SOLAR ATMOSPHERE[1]

By HENRY NORRIS RUSSELL[2]

ABSTRACT

Ionization in the solar atmosphere.—The absorption of photospheric radiation by atoms in the sun's atmosphere tends to increase the degree of ionization, both directly, by shifting an electron into a position from which its removal is easier, and indirectly, when enhanced lines are absorbed, by getting the ionized atoms into states in which they are probably less likely to combine with electrons. Both these influences operate strongly in the case of barium and weakly, if at all, for sodium. This probably explains why barium is much more highly ionized in the sun than sodium, although their ionization potentials are the same. The case of lithium, which is also ionized to an abnormally high degree, cannot be explained in this way. The greater width of the lines, due to the thermal agitation of so light an atom, may help to solve the problem.

Barium.—The lines of this element which are present, or doubtful, in the solar spectrum are tabulated. All the strong lines arise from the ionized atom. A few faint lines of the neutral atom are probably present.

Rubidium.—Three enhanced lines of this metal are present, though faint, in the normal solar spectrum.

Manganese.—The ionization potential 7.6 volts agrees well with the value predicted from the behavior of its lines in the solar spectrum.

The following remarks may be regarded as extensions of the writer's recent paper, "The Theory of Ionization and the Sun-Spot Spectrum."[3] The ideas on which they are based grew largely out of conversations between the writer and Professors K. T. Compton, of Princeton, and F. A. Saunders, of Harvard, to both of whom he desires to acknowledge his indebtedness.

1. *Barium.*—The spectroscopic behavior of this element indicates that it is ionized to a much higher degree in the solar atmosphere, and even in the spots, than would be expected from its ionization potential, which is substantially identical with that of sodium.

The statement in the previous paper, that none of the lines of the neutral barium atom appear in the solar spectrum, is incorrect, for the second line of the principal series at λ 3071 is faintly present, although the first (at λ 5535) is absent or excessively faint, and

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[1] *Contributions from the Mount Wilson Observatory*, No. 236.

[2] Research Associate of the Mount Wilson Observatory.

[3] *Mt. Wilson Contr.*, No. 225; *Astrophysical Journal*, **55**, 119, 1922.

King[1] records that a number of lines belonging to the fundamental series of triplets,[2] which are very diffuse in the arc, but sharp in the vacuum furnace, are present in the sun, though very faint. Table I gives a list of all those barium lines which can be identified with tolerable certainty in the solar spectrum. The first three columns

TABLE I

LINES OF BARIUM IN THE SOLAR SPECTRUM

ARC SPECTRUM (KING)			SOLAR SPECTRUM (ROWLAND)				
I. A.	Int.	Class	λ	Origin	Int.	Δλ	Notes
3071.592.......	100 R	III	1.662	Ba	00	−0.044	(1)
3891.788.......	50	V	1.918	....	0	− .012	(2)
3995.633.......	30	III	5.769	Ba?	000	− .010	(3)
4130.683.......	80	V	0.804	Ba	2	− .036	(2)
4166.027.......	20	V	6.161	Ba	0	− .018	(2)
4524.946.......	35	V	5.110	....	0	− .004	(4)
4554.038.......	1000 R	II	4.211	Ba	8	+ .003	(5)
4636.333.......	15 N	IV	6.501	....	0	− .007	....
4934.099.......	700 R	II	4.214 4.277	Ba	3 4	− .027	(5)
5267.013.......	20	IV?	7.203	....	000	+ .005	....
5853.699.......	200	III	3.902	Ba?	5	− .009	....
6141.760.......	600 R	III	1.938	Fe, Ba	7	− .032	....
6496.902.......	600 R	III	7.128	Fe	4	+0.013	(6)
	Doubtful Lines						
3993.395.......	80	II	3.616	Ba?	00 N	+0.065	(3)
4132.444.......	20	I	2.560	Ba?	0	− .041	....
4283.111.......	100	II	3.414	Ba?	00 N	+ .141	....
4350.375.......	80	II	0.550	Ba?	00 N	+ .012	....
4505.936.......	40	II	6.259	Ba?	00 N	+ .156	....
4573.881.......	40	II	3.960	Ba?	000	− .091	....
4899.970.......	35	IV?	0.095	Ti, La	2	(− .055)	(4) (7)
6019.505.......	100	II	9.596	Ba?	000	− .127	....
6110.808.......	300 R	II	1.005	....	0000	+ .064	....
6595.351.......	125	I	5.590	....	0000	+0.010	....

NOTES

1. Second member of the principal series of neutral atom 1s–3p.
2. Second member of diffuse series of enhanced lines 2P–4D.
3. Part of first member of fundamental series of neutral atom 3d–4f.
4. First pair of sharp series of enhanced lines 2P–3S.
5. First pair of principal series of enhanced lines 1S–2P.
6. Rowland gives this as an iron line: but the line in question is at 6496.462 (I.A.) according to Burns. The solar line is probably due to barium.
7. The barium line, if present, is masked.

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[1] *Mt. Wilson Contr.*, No. 150; *Astrophysical Journal*, **48**, 13–34, 1918.

[2] Saunders, *Astrophysical Journal*, **51**, 25, 1920.

give the wave-length in I.A. from the measures of King or Schmitz, the intensity in the arc, and the temperature class; the next three, Rowland's wave-length, origin, and intensity; and the next, the difference $\Delta\lambda$ between Rowland's wave-length and the laboratory value after the application of St. John's value of the systematic correction.

All the high-temperature lines of King's classes IV and V, which are of intensity greater than 20 in the arc, appear in the solar spectrum, while numerous very strong low-temperature lines of classes I and II and intensities from 200 to 1000, are entirely absent. Most of the lines which are certainly present are well-known enhanced lines. The strong lines at 5853, 6141, and 6496 constitute, according to Saunders,[1] the first member of the diffuse series of enhanced lines, and are analogous to λλ 8662, 8542, 8498 of calcium, but reversed in order of wave-length. They are unaffected in the spot spectrum. The enhanced lines of the sharp series appear to be faintly present, and not absent, as stated earlier.

It appears, therefore, that neutral atoms of barium are probably present in the solar atmosphere, in a proportion which, though very small, is not insensible, as was stated in the preceding paper. The conclusion that barium is very highly ionized in the reversing layer, and even in spots, is confirmed by the re-examination of the data.

The conspicuous difference between its behavior and that of sodium remains to be explained. The solution of the puzzle probably lies in the fact that two ionizing agencies are certainly at work in the solar atmosphere—the temperature of the medium, and the radiation poured through it from the hotter photosphere below. Only the first is taken into consideration in the equations of Nernst and the theory of Saha, and, if it alone were active, there would be very strong reason to expect that the percentage of ionized atoms would be the same for the two elements. It is easy to see, however, that the second must create a difference of the sort which is observed.

Consider a barium atom which has just become ionized. Its first (or most easily removable) outer electron is lost; the second will be in the normal state of least energy content, denoted by 1S. In this condition it will immediately recombine with an electron which meets it under suitable conditions. But, if photospheric radiation (λλ 4934, 4554) falls upon it, it will absorb energy and the second electron will be lifted into one of the pair of positions 2P. If, after this, the charged atom meets a free electron, it is not so certain what will happen. It may be that it cannot recombine at all unless the second electron is in the normal position. If it can do so, the energy corresponding to the shifting of the second electron and the binding of the first one would have to be radiated. If this was done in a single quantum, the frequency would correspond to a "combination" between terms of the series of ordinary and enhanced lines, and such combinations have never been observed. Leaving aside the more delicate question whether the atom could emit two quanta at once from different electrons, it appears probable in any event that the $Ba+$ atom with its second electron in the 2P state would be *less likely* to recombine than one in the 1S state. Moreover, it will be decidedly easier to ionize a second time, for the energy required to remove the second electron is less (7.15 volts instead of 9.86, after absorption of λ 4554). There is also the probability to be considered that the absorption of a second quantum of radiation may raise the second electron to the state 3D, to which the considerations just urged will apply with even greater force.

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[1] Unpublished.

The absorption of photospheric radiation by the ionized atoms of barium must therefore discourage recombination and encourage further ionization. Similar absorption by the neutral atoms will assist their ionization, so that, on the whole, the radiation must have a powerful ionizing influence.

For sodium, this influence must be very much smaller, and that for two reasons. In the first place, the enhanced lines of sodium, or at least the fundamental ones, lie in the extreme ultra-violet, beyond λ 377,[1] where the radiation of the photosphere is probably insensible, so that the sodium atom, once ionized, is not further disturbed by sunlight. Secondly, the principal lines of the neutral atom (the D lines) are very strong in the solar spectrum, with wings which indicate that they are absorbed to a great depth. Sodium atoms in the outer parts of the solar atmosphere are therefore exposed to a much enfeebled radiation of just those wavelengths which would be effective in aiding ionization.

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[1] Millikan, *Proceedings of the National Academy of Sciences*, 7, 290, 1921.

For the corresponding lines of barium, on the contrary, the absorption is exceedingly small, and neutral atoms in the upper atmosphere are exposed to the full influence of the radiation.

When all these factors are considered, it is no longer surprising that barium is more highly ionized than sodium, in spite of the equality of their ionization potentials.

Whether the influence of radiation is competent to produce as great an effect as that which is observed can be determined only when a quantitative theory of its action has been developed. It may be noticed, however, that the 3D–2P lines of $Ba+$ are almost as strong in the solar spectrum as the pair 1S–2P, which indicates that a large fraction of all the singly ionized barium atoms in the reversing layer are in the states of "abnormally" large energy content.

The principle here sketched is evidently generally applicable. An element which has strong enhanced lines in the region where the continuous spectrum of the sun (or, for that matter, of any star) is strong, will behave as if it was relatively easier to ionize than one which does not possess such lines. The behavior of strontium, which resembles barium, but is less conspicuously "anomalous," is thus explained.

Most of the elements which are prominent in the solar spectrum possess such enhanced lines, with the exception of the alkali metals. The spectroscopic method suggested in the former paper should, therefore, give relatively high values for the ionization potentials of these metals, while it is probable that the errors for the other elements will be smaller. The alkaline earths, whose enhanced lines are very favorably situated, will probably come out with ionization potentials too low.

2. *Manganese.*—Saha[1] quotes the value 7.6 volts for the ionization potential, derived from unpublished studies by Catalan upon the series in its spectrum. This agrees satisfactorily with the spectroscopic evidence which puts its ionization potential rather more than halfway from *Ca* (6.1) to *Zn* (9.4).

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[1] *Nature*, 107, 683, 1921.

3. *Rubidium.*—Saunders[1] notes that the strongest of the enhanced lines given by Kayser[2] appear to be present, though faint, in the solar spectrum—the wave-lengths on Rowland's scale being given in Table II.

TABLE II

ENHANCED LINES OF RUBIDIUM

Spark (Exner and Haschek)		Sun (Rowland)	
λ	Int.	λ	Int.
3940.62	20	Absent	....
4104.49	10	4104.459	00
4244.50	30	4244.500	00
4294.08	10	4294.077	000

The evidence for the presence of this metal in the sun is therefore strengthened. The principal enhanced lines lie probably in the far ultra-violet.

None of the stronger enhanced lines of the other alkali metals, as given in Kayser's *Handbuch*, appear to be present in Rowland's tables.

4. *Lithium.*—The absence of lithium lines from the solar spectrum, and their faintness in spots, still presents much difficulty. Lithium behaves almost exactly like rubidium, though its ionization potential is 5.37 volts, and that of the latter 4.16.

We cannot invoke any absorption of enhanced lines here, for none have even been discovered.[3] Compton suggests[4] that one factor in the problem is the small atomic weight. This will cause the mean atomic velocity of temperature agitation to be greater than for other elements (about three and one-half times that for rubidium), and so broaden out all its lines, so that the atoms are able to take hold (so to speak) of a correspondingly greater range of wave-length in the photospheric radiation, and use it in assisting their ionization.

What other factors may enter remains for future discovery.

MT. WILSON OBSERVATORY
February 22, 1922

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[1] Not previously published.

[2] *Handbuch der Spectroskopie*, 6, 350.

[3] Millikan, *op. cit.*, p. 292.

[4] Unpublished.

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THE EDITORS

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PARTIAL EXPLANATION, BY WAVE-LENGTHS, OF THE K-TERM IN THE B-TYPES

By SEBASTIAN ALBRECHT

ABSTRACT

K-term of the radial velocity of Class B stars.—Whether the K-term is due to an expanding universe, to causes within the stars or to the Einstein effect, it is of importance to determine *the part due to the use of inaccurate normal wave-lengths* in the reductions. For most of the lines upon which the B-type velocities are based, such as those of Si and He, sufficiently accurate wave-lengths are not available; but in the case of twenty O and N lines, new wave-lengths by Clark are found to be systematically 0.063 A longer than the adopted normals. A discussion of the available portions of the data on which the K-term is based, indicates that the new wave-lengths for this one group of lines alone would reduce the K-term, which is about 4 km, by about 0.3 km for the entire B class and by about 0.8 km for classes B0–B2.

Test of the relative accuracy of groups of wave-length normals.—Since the spectra of different classes of stars are radically different, it is *suggested* that a comparison of the mean wave-lengths for the lines which these classes have in common might give evidence as to the relative accuracy of the principal group of normals involved in each case. For instance the wave-lengths of types B0–B2 are based more on O, N, and Si lines than those of types B3–B5 and they are systematically 0.015 A shorter.

In 1911 Campbell[1] found that, after the radial component of the sun's motion in space had been deducted from the radial velocity of each star, the remaining portions of the radial velocities were not at random but in every part of the sky showed a systematic average residual directed away from the observer. This residual, which has become known as the K-term, appeared to be essentially the same for all limited areas of the sky, but to differ according to the spectral class of the star. Interpreted as a phenomenon of stellar motion, it means that the universe, represented by the stars employed by Campbell, is an expanding one, the rate of expansion depending upon the spectral class. Thus the stars of types B and M are systematically moving away in all directions from the position occupied at any instant by the solar system with a speed of about 4 km per second. The A and K types are moving outward with speeds of about 1 and 2 km per second respectively, while the F and G types show a K-term which is essentially zero.

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[1] *Lick Observatory Bulletins*, **6**, 107 and 127, 1911.

However, the K-term could be produced, as was also pointed out by Campbell, by the use of wave-lengths which are on the average too low. Thus if it could be assumed that the wave-lengths which were employed in the reductions for radial velocity for the B types were too small by about 0.06 A, then the entire K-term for these types would disappear. Campbell was inclined toward the view that such changes in the wave-lengths may be due to causes within the stars, such as high pressures, absorption through a wide range of depth in the stellar atmospheres and, with particular reference to helium, conditions in the stars such as to augment considerably the red components of lines which occur in close pairs. Freundlich[1] suggested that, due to the Einstein effect, all stellar lines are displaced toward the longer wave-lengths relatively to terrestrial comparison spectra. Such displacements, which would merge with displacements due to radial velocity and reappear in the K-term, would differ for stars of the different spectral classes due to mean differences in their masses. Although these factors must not be disregarded, nevertheless the laboratory values themselves require close scrutiny.

At about the same time that the K-term was found, attention was called by the writer to the unsatisfactory state of our knowledge in regard to the laboratory values of the wave-lengths upon which the radial velocities for the B-type depend.[2] This is especially true for the three prominent and important silicon lines λ 4552, λ 4567, and λ 4574, for which the two most accurate laboratory determinations differ by an average of 0.09 tenth meters, corresponding to a systematic difference of 6.0 km in the

[1] *Astronomische Nachrichten*, **202**, 17, 1915.

[2] *Publications of the Astronomical and Astrophysical Society of America*, **2**, 71, 1911.

radial velocities derived from them. There has been only one response, I believe, in the ten years which have elapsed, to the appeal which was made for additional laboratory determinations. Mr. Clark,[1] under the direction of Professor Fowler, has redetermined the wave-lengths for the oxygen and nitrogen lines. The effect of the new wave-lengths upon the radial velocities and the K-term can be approximated.

TABLE I

λ Clark (Rowland System)	λ Normal (F. and A.)	λ Clark − λ Normal	Change in Radial Velocity	Element
		A	km	
4317.327	.272	+0.055	−3.8	O
19.814	.762	+ .052	−3.6	O
45.738	.677	+ .061	−4.2	O
48.246	.134	+ .112	−7.7	N
49.602	.541	+ .061	−4.2	O
51.443	.495	− .052	+3.6	O
67.074	.012	+ .062	−4.3	O
4415.057	.076	− .019	+1.3	O
17.143	.121	+ .022	−1.5	O
4447.205	.163	+ .042	−2.8	N, O
4591.158	.066	+ .092	−6.0	O
96.365	.291	+ .074	−4.8	O
4621.582	.548	+ .034	−2.2	N
30.729	.703	+ .026	−1.7	N
39.043	.937	+ .106	−6.9	O
42.006	.886	+ .120	−7.8	O
49.327	.250	+ .077	−5.0	O
51.032	.925	+ .107	−6.9	N, O
61.829	.728	+ .101	−6.5	O
4676.426	.291	+ .135	−8.7	O
Means (20 lines)	....	+0.063	−4.2	....

Table I, which is self-explanatory, contains such of the oxygen and nitrogen lines as were included in the reductions for radial velocity. For the 20 lines listed the wave-lengths by Clark are on the average higher by 0.063 A, which corresponds to a change of −4.2 km in the radial velocities derived from them.

The details of the radial velocities from which Campbell derived the K-term have not been published, so that it is not possible to determine the exact part played by any particular group of lines in the reduction of his 225 B-type stars. However, this can be very roughly approximated with the aid of two partial sets of data for which such details are available, namely, the measures of 20 Orion-type stars by Frost and Adams[1] and the B-type stars contained in Volume IX of the *Publications of the Lick Observatory*. For these data the changes in the radial velocities produced by the substitution of the wave-lengths of Clark in place of the normal wave-lengths which were employed are summarized in Tables II and III. Tables IV and V show, for the smaller subdivisions of Class B, the relative representation of the different groups of lines in the measures. As far as can be judged from Tables II to VI the inference seems to follow that the new wave-lengths of Clark for oxygen and nitrogen will reduce the K-term for the entire Class B by about 0.3 km, and for B0–B2 by about 0.7 or 0.8 km.

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[1] *Astrophysical Journal*, 40, 332, 1914.

TABLE II

CHANGES IN RADIAL VELOCITY PRODUCED BY WAVE-LENGTHS OF CLARK. MEASURES BY FROST AND ADAMS

Star	Type	Change for O and N Lines		Change for Star		Relative Representation of the O and N Lines in the Measures (in per cent)
		km	Plates	km	Plates	
ε Orionis	B0	−6.9	1	−0.1	7	22
ζ Orionis	B0	−1.5	3	−0.2	7	13
κ Orionis	B0	−1.4	7	−0.2	10	15
ζ Persei	B1	−2.5	7	−0.7	7	31
β Can. Maj.	B1	−2.2	5	−1.0	5	42
ε Can. Maj.	B1	−2.2	4	−0.7	4	33
γ Pegasi	B2	−0.4	12	−0.2	12	22
ζ Pegasi	B2	−0.5	3	−0.1	6	8
γ Orionis	B2	−2.9	6	−0.4	10	9
102 Herculis	B2	−0.1	2	0.0	5	6
ι Herculis	B3	+1.3	1	0.0	6	1
η Lyrae	B3	−1.5	1	0.0	6	2
ζ Draconis	B5	−1.5	2	−0.1	6	5
ε Delphini	B5	−0.1	2	0.0	4	8
β Orionis	B8p	−1.5	1	0.0	24	0

It does not seem unlikely that additional portions of the K-term may have their origin in the normal wave-lengths of other groups of lines. In this connection it may not be amiss to recall

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[1] *Publications of the Yerkes Observatory*, 2, 143, 1904.

TABLE III

CHANGES IN RADIAL VELOCITY PRODUCED BY WAVE-LENGTHS OF CLARK MEASURES IN *Publications of Lick Observatory*, 9

Star	Type	Change for O and N Lines	Change for Star	Relative Representation of the O and N Lines in the Measures	Measured by
		km	km	Percentage	
τ Scorpii	B0	−2.0	−0.9	44	A
		−2.0	−1.2	56	P
β Crucis	B1	−1.7	−0.8	45	A
		−1.9	−1.0	53	P
α Pyxidis	B2	−2.1	−1.0	46	A
α Lupi	B2	−1.8	−1.1	60	A
		−1.9	−1.0	52	P
α Pavonis	B3	−1.6	−0.3	15	A

A = S. Albrecht P = H. K. Palmer

TABLE IV

FROM MEASURES BY FROST AND ADAMS

Type	No. of Stars	Hγ	Mg	He	Si	O and N	Metals
		Percentage	Percentage	Percentage	Percentage	Percentage	Percentage
B0	3	15	4	58	10	11	0
B1	3	3	3	23	33	36	1
B2	4	5	14	42	26	11	1
B3	2	2	30	54	12	1	0
B5	5	12	35	47	4	2	1
B8	2	8	29	28	1	0	35
B0–B8	19	8	19	43	14	11	5

TABLE V

FROM *Publications of the Lick Observatory*, 9

Type	No. of Stars	Hγ	Mg	He	O and N	C	Metals
B0	1	5	6	16	50	5	18
B1	1	15	5	32	33	10	5
B2	6	33	4	39	13	9	3
B3	9	30	14	42	1	12	1
B5	2	0	32	68	0	0	0
B8	2	64	14	14	0	6	*
B0–B9	21	29	12	39	8	9	2

* A number of unidentified lines were not included in the percentages for B8.

that systematic differences between Lick and Bonn radial velocities[1] for types F–M, of the same order of magnitude as the K-term for these types, have been found to be due to differences in the normal wave-lengths which were employed in the reductions. The normal wave-lengths for the lines in Class B are perhaps not as accurately known as those for types F–M. I wish again to urge the desirability of additional laboratory study of the other groups of lines which occur in Class B stars, and especially those of silicon and helium. In view of the importance of the problem and the apparent difficulties with the laboratory spectra, a certain amount of duplication would not be wasted effort.

TABLE VI

DATA FROM WHICH THE K-TERM WAS DETERMINED

Type	No. of Stars	Part of Total Number (in per cent)	
B0	13	6	24
B1	17	7	
B2	25	11	
B3	72	32	56
B4	2	1	
B5	51	22	
B8	22	10	20
B9	21	10	
A0*	2	1	
B–A	225	100	

* Two of Campbell's stars seem to have been reclassified as A0 at Harvard.

If Tables IV and V may be regarded as approximately representative for the B types, they may also serve as a basis for a somewhat coarser grouping of the subdivisions of Class B for certain statistical purposes. B. Boss pointed out that the proper motions of stars of Classes B8–B9 more nearly resemble those of A type than those of early B type, and that therefore the B8–B9 stars should be grouped with early A types rather than with early B types. These results were confirmed by Campbell, and my Table IV is also in accord with this conclusion, the abruptly large representation of the metallic lines being the principal distinguishing characteristic. Additional subdivisions indicated are B0–B2 and B3–B5. In the former silicon and oxygen are strongly represented in the measures and magnesium only slightly, while in the latter the reverse is true.

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[1] *Astrophysical Journal*, 40, 475, 1914.

The low values of the normal wave-lengths for the oxygen and nitrogen lines contribute toward the progression in K which is indicated within the B types. Campbell's 225 stars are probably inadequate, under present limitations, for accurate solutions for subdivisions B0–B2, B3–B5, and B8–B9. Campbell (*op. cit.*, p. 123) finds K = +4.7 km per second for stars of Class B0–B5 and only +4.1 km for the entire class B0–B9. His Class B0–B5 comprises 80 per cent (see Table VI) and his Class B8–B9 only 20 per cent of his list. Campbell also finds K = +1.62 km for 222 stars of Classes B8–A9 and K = +0.95 km for 177 stars of Classes A0–A9 alone. On the basis of proportionality the former results would give approximately K = +2 km, while the latter results would give K = +4 km for classes B8–B9. Perhaps a K-term of about +3 km for B8–B9 will fairly well represent the actual facts for the data concerned. In view of the strong representation of the oxygen and nitrogen lines in B0–B2 and their practical absence in the measures of stars B3–B5, K is probably greater in the former than in the latter subdivisions.

The observed wave-lengths for the stellar lines are capable of contributing, so to speak, circumstantial evidence toward the relative correctness of the adopted normal wave-lengths for different groups of lines. A general discussion of this nature will be more worth while after the laboratory has contributed more nearly its full share toward this problem. For the present I shall be content with merely indicating how low values of the normal wave-lengths for the oxygen and nitrogen lines will impress a corresponding effect upon the observed stellar wave-lengths. The subdivisions B0–B2 and B3–B5 are suitable for this purpose, because the oxygen and nitrogen lines are strongly represented in the former and nearly absent in the latter. If the normal wave-lengths for the oxygen and nitrogen lines, constituting nearly 20 per cent of the lines actually employed in B0–B2, averaged 0.056 A too low, the radial velocities for these lines will average 3.7 km too high, making the mean radial velocities for the stars about 0.74 km too high. In the residuals of velocity the oxygen and nitrogen lines will show up as 2.96 km too high, and the remaining 80 per cent of the lines as 0.74 km too low. Thus the observed wave-lengths for the oxygen and nitrogen lines will be brought up four-fifths of the way toward where they should be, while the other 80 per cent of the lines will have their observed wave-lengths lowered from where they should be by an amount equal to one-fifth of the average error in the adopted normal wave-lengths for oxygen and nitrogen. Thus all the observed wave-lengths on the plates will finally be 0.011 A too low. Therefore, considering only the effects due to the oxygen and nitrogen lines, all the observed wave-lengths in classes B0–B2 should, on the average, be 0.011 A lower than in classes B3–B5. Table VII may be of interest in this connection, even though the wave-lengths given are not strictly definitive and the effects of errors in the normal wave-lengths of other groups of lines are also involved. The subscripts represent provisionally adopted weights, based partly upon the number of plates and partly upon the number of stars. Of the seventeen comparisons listed, fourteen give the wave-length smaller in B0–B2 than in B3–B5 and only two give it as larger. The mean difference in wave-length is 0.015 A, not all of which is necessarily due to the oxygen and nitrogen lines.

TABLE VII

OBSERVED WAVE-LENGTHS IN CLASSES B0–B2 AND B3–B5

LINE	NOR. MILLS λ 4500 CENTRAL		FROST AND ADAMS λ 4500 CENTRAL		SO. MILLS λ 4340 CENTRAL		β CAN. MAJ. λ 4500 CENTRAL
	B0–B2	B3–B5	B0–B2	B3–B5	B0–B2	B3–B5	B1
4267.3 C.......	....	....	....	....	0.260_{14}	0.278_{18}	....
4340.6 Hγ......	0.618_{30}	$.643_{16}$	0.622_{30}	0.642_{16}	$.625_{15}$	$.632_{18}$	$.609_{29}$
4388.1 He......	$.073_{43}$	$.109_{6}$	$.094_{37}$	$.099_{19}$	$.109_{15}$	$.109_{18}$	$.082_{33}$
4437.7 He......	....	....	$.713_{26}$	$.726_{13}$	$.727_{12}$	$.725_{15}$	$.721_{22}$
4471.6 He......	$.664_{46}$	$.685_{14}$	$.668_{39}$	$.689_{22}$	$.684_{15}$	$.707_{18}$	$.672_{34}$
4481.4 Mg......	$.365_{42}$	$.375_{13}$	$.376_{33}$	$.397_{22}$	$.368_{15}$	$.376_{18}$	$.362_{34}$
4552.7 Si	$.752_{47}$	$.798_{8}$	$.769_{34}$	$.752_{8}$	....	....	$.753_{34}$

The results in Tables II–V are probably fairly representative for the dispersion of three prisms. For several reasons, among them the broad character of the spectrum lines in many B-type stars, a considerable number of the radial velocities are based on the lower dispersions of one and two prisms. The effects of systematic differences between results obtained with dispersions of one, two, and three prisms will also have to be evaluated. The details required for this purpose have not been published. However, unless my memory is in error on this point, the radial velocities obtained with the lower dispersions showed a systematic tendency toward larger positive values than those obtained with three prisms. Any uneliminated portions of such systematic differences will directly contribute toward the K-term.

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Dudley Observatory
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EFFECTIVE WAVE-LENGTHS OF 129 STARS WITHIN 0°5 OF THE NORTH POLE AND OF THE Σ COMPANION OF POLARIS[1]

BY EJNAR HERTZSPRUNG

ABSTRACT

Effective wave-lengths, color indices, and magnitudes of 129 stars within 0°5 of the North Pole.—The results of measurements of about 500 images on 12 plates (Lumière Σ) taken in 1912 with the 60-inch reflector and a grating whose constant was 6 mm, are given in Table II. In determining the effective wave-length λ_{eff}, corrections were made (1) for diameters differing from the normal diameter 0.10 mm and (2) to reduce to zenith transmission. For each star the effective color index I_λ was obtained from λ_{eff} by use of the formula: $I_\lambda = 0.005(\lambda_{\text{eff}} - 4216\text{ A}) = 19.721(\lambda_{\text{eff}} - 1.069\text{ mm})$. The star, *Harvard Annals*, **48**, No. 85, is probably wrongly included in the series of red stars of the North Polar Sequence. For the Σ *companion to Polaris*, I_λ was found equal to $+0.11 \pm 0.07$, which corresponds to a spectrum of type A0 to A5.

Relation between effective color-index and the color-index of Seares.—For 76 stars both I_λ and I_{MW} (the difference between photovisual and photographic magnitudes) were available for comparison. It was found that $I_\lambda = (0.90 \pm 0.05) I_{MW} - (0.10 \pm 0.05)$, a very satisfactory result.

Relation between photographic magnitude and diameter of image was found to be: $m = \text{Const.} - 34d + 35d^2$, for d between 0.05 and 0.30 mm.

During my stay at Mount Wilson in 1912 the plates of the North Polar region listed in Table I were taken with the 60-inch reflector for the determination of effective wave-lengths.[2] Lumière Σ plates were used. The focal length of the instrument is 7606 mm and the grating constant was exactly 6 mm.

The co-ordinates of the center of the field are given in the system of *Harvard Annals*, **48**, 21, Table VI, sixteenth and seventeenth columns. The efficiency of each plate is indicated by the photographic magnitude of stars whose central image has a diameter of one-tenth of a millimeter. The total number of images measured is somewhat greater than that used in the present note because some stars showing only very faint images were finally rejected. The effective wave-lengths of hazy images were given half-weight. The mean weight-factor of a plate, which is given in the last column of Table I, may therefore be taken as a measure of the mean quality of the images. The quality 0.50 thus means that all images of the plate were given half-weight.

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[1] *Contributions from the Mount Wilson Observatory*, No. 231.

[2] Compare *Mt. Wilson Contr.*, Nos. 100, 101; *Astrophysical Journal*, **42**, 92, 111, 1915.

TABLE I

PHOTOGRAPHS OF THE NORTH POLE

Plate No.	Date 1912	Aperture (inches)	Sidereal Time at Middle of Exposure	Exposure Time (min.)	Co-ordinates of Center of Field		Efficiency of Plate	Number of Images Measured	Quality of Images
					x	y			
4	July 19	60	$20^h\ 5^m$	60	−480″	+ 70″	$14^{m}8$	32	0.64
5		40	22 23	95	−520	+ 30	14.6	108	.70
6		40	23 26	9.5	−510	+ 50	12.0	15	.83
10	July 20	40	21 1	9.5 3	−180 −350	+ 50	11.9 10.9	29	.83
11		40	22 36	95	−190	+ 50	14.6	112	.69
14	July 21	40	18 16	31.3	+ 10	+140	13.3	44	.50
15		40	19 15	30	− 40	+ 90	13.7	34	.56
16		40	19 53	10, 5 1, 0.3	−60, +20 +120, +190	+ 60	12.7, 11.7 10.7, 9.7	27	.52
17		60	21 48	60	−200	+ 30	14.9	23	.50
204	September 14	40	21 7	6	−120	− 10	11.6	14	.79
205		40	21 43	60	− 80	− 50	13.8	44	.56
206		40	22 29	19	− 60	− 70	12.7	29	0.83

The diameter of the field measured was 1150″ when the full aperture was used, and 2500″ when the mirror was diaphragmed down to 40 inches. The plates were measured and reduced at Potsdam. At the same time a series of plates of the Pleiades has been discussed, which afford 9972 effective wave-lengths. This last material has been used for a more thorough determination of the variation of the effective wave-length with the intensity of the image. The effective wave-lengths were measured in millimeters by moving the plateholder with a micrometer screw and reading to $\frac{1}{2}\ \mu$. The diameter of the central image of the star was taken as a measure of the intensity of the spectra of first order. As normal strength that shown when the diameter of the central image is 0.1 mm or 100 μ was chosen. For other diameters of the central image the following corrections to the normal diameter were found:

Diameter of central image $d=$	0.05	0.06	0.07	0.08	0.09	0.10 mm
Correction to $d=0.10$ mm	+13	+10	+7	+4	+2	+0 μ

0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19−0.24 mm
−2	−3	−4	−5	−6	−6	−7	−7	−7 μ

No conspicuous difference in the corrections for stars of different effective wave-lengths was noted.

These corrections were also used in reducing the effective wave-lengths given in the present note. The correction required to reduce the effective wave-lengths to the zenith owing to selective extinction of light in the earth's atmosphere amounts to −23 A or −6 μ for the North Polar region as seen from Mount Wilson.

To convert the effective wave-lengths expressed in millimeters into Angstrom units they must be multiplied by 3944.2. It will be more convenient for comparison with determinations of the difference between photographic and visual magnitudes, however, to transform the effective wave-length measured in mm directly into the equivalent values of color index.

I have found that a difference of 1^{M} in the color index in the system of I_H of the Göttingen Actinometry, Part B, corresponds nearly to a difference of 200 A in the effective wave-length. By adopting this and comparing the color indices I_H of stars occurring in the Göttingen Actinometry with their effective wave-lengths on other plates taken with the 60-inch reflector it was found that $I_H=0$ corresponds to an effective wave-length of 1.069 mm or 4216 A. Hence, indicating the color index calculated from the effective wave-length by I_λ, we have

$$I_\lambda=\tfrac{1}{200}(\lambda_{\text{eff}}-4216\text{ A})=19.721(\lambda_{\text{eff}}-1.069\text{ mm})$$

The accuracy of the effective wave-length varies with the intensity of the image. Different weights were therefore assigned to the effective wave-lengths, according to the diameter of the central image. These weights were decreased for images which were not of normal strength, because of the uncertainty of the reduction of the effective wave-length to $d=0.10$ mm, which may vary with the quality of the image. The following weights of I_λ were finally adopted for different values of the diameter d of the central image.

$d=$	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21	0.22	0.23	0.24 mm
$1/\lambda_2(I_\lambda)=1/(\text{m.e.})^2$	2	4	8	18	28	38	48	56	58	52	44	36	28	22	16	12	8	6	4	2

For example, for $d=0.19$ mm, the adopted weight is 16, corresponding to a mean error in the color index I_λ of $\pm\sqrt{1/16}=\pm 0.25$. As stated above, for bad (hazy) images these weights were divided by 2. In fact the North Polar plates here employed are perhaps the poorest I took at Mount Wilson.

The results are given in Table II. The first column contains the number of the star taken from the *Harvard Annals*, **48**, 21, Table VI, and the second, the number of the North Polar sequence. The third column gives the photographic magnitude as derived from my estimates of the diameter of the central image. These magnitudes, according to which the stars have been arranged in the table, are only approximate. No correction for distance from the center of the field has been applied. The scale of Seares[1] was taken as a standard. The following relation between magnitude and diameter d of the central image was found.

$d=$ (mm)	m+plate constant=
0.04	16.0
0.05	15.9
0.06	15.8
0.07	15.5
0.08	15.2
0.09	14.8
0.10	14.6
0.11	14.3
0.12	14.0
0.13	13.8
0.14	13.6
0.15	13.3
0.16	13.1
0.17	12.9
0.18	12.7
0.19	12.5
0.20	12.2
0.21	12.0
0.22	11.9
0.23	11.6
0.24	11.5
0.25	11.4
0.26	11.2
0.27	11.0
0.28	10.9
0.29	10.7

The fourth column contains the color index I_λ as derived from the effective wave-length, and the fifth column, its weight, i.e., the reciprocal square of the mean error. The sum of the weights of I_λ for all 129 stars is 10,281. The sixth column gives the number of images used, respectively, for the estimation of magnitude and for the determination of effective wage-length.

Of special interest is a comparison of I_λ with the color index of Seares[2] which is defined as the difference between the photographic and the photovisual magnitude

$$I_{MW}=m_{Pg}-m_{Pv}.$$

This comparison has been made in the following way. In Table II there are 77 stars for which both I_λ and I_{MW} have been deter-

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[1] *Mount Wilson Contr.*, No. 97, Table IX; *Astrophysical Journal*, 41, 206, 1915.

[2] *Loc. cit.*

TABLE II

COLORS OF THE NORTH POLAR STARS

No.		m_{pg}	I_λ	$p(I_\lambda)$	n_m, n_λ
H.A. 48	N.P.S.				
420	10	9.48	+ .03	66	8,5
356		10.37	− .03	30	3,2
101	13	10.59	+ .21	173	7,6
386	14	10.89	+ .35	240	12,10
184	7*r*	11.03	+ .93	293	12,10
350		11.09	+ .22	252	13,11
195	8*r*	11.10	+ .83	24	2,2
227	5*s*	11.23	+ .90	300	14,12
197	15	11.26	+ .17	271	12,11
419		11.40	+ .40	44	3,3
309		11.44	+ .27	144	8,7
75	6*s*	11.46	+ .65	255	12,11
62		11.57	+ .70	71	3,3
408	16	11.63	+ .15	209	9,8
282	17	11.85	+ .61	76	4,4
430		12.13	+ .71	57	3,3
67	9*r*	12.2	+1.28	6	1,1
320	18	12.32	+ .21	183	12,10
85	10*r*	12.62	+ .30	174	9,9
225	7*s*	12.66	+ .39	206	14,11
268	19	12.75	+ .25	194	10,9
494		12.83	+ .66	107	6,5
57		12.93	+ .43	124	9,8
302	20	12.94	+ .23	179	10,9
362		13.05	+ .54	158	8,7
219	11*r*	13.20	+1.11	201	10,9
427		13.24	+ .96	91	7,5
262		13.4	+ .56	32	2,2
234	23	13.44	+ .35	190	8,8
479		13.44	+ .51	88	5,5
489		13.45	+ .14	98	4,4
451		13.48	+ .45	83	5.5
292	22	13.52	+ .44	144	6,6
174		13.54	+ .52	105	5,5
248	21	13.57	+ .79	144	6,6
389		13.70	+ .60	116	5,5
151		13.79	+1.01	194	7,7
103		13.8	+ .81	58	1,1
436		13.82	+ .39	59	4,4
172		13.84	+1.21	144	5,4

TABLE II—*Continued*

No.		m_{pg}	I_λ	$p(I_\lambda)$	n_m, n_λ
H.A. 48	N.P.S.				
426		13.90	+ .32	95	4,4
25		14.0	+1.00	56	1,1
34		14.0	+ .05	28	1,1
300	12*r*	14.00	+1.17	149	5,5
279	24	14.04	+ .36	116	5,5
345		14.08	+1.50	141	5,5
90		14.12	+1.23	129	5,4
127		14.13	+ .66	168	6,4
243		14.15	+ .32	52	2,2
378		14.15	+1.07	116	4,3
146		14.17	+ .55	108	3,3
28		14.3	+ .15	24	1,1
33		14.3	+ .89	24	1,1
329		14.3	+ .88	115	4,4
134		14.33	+ .41	141	6,6
263	25	14.35	+ .30	117	6,6
319		14.38	+ .62	26	4,2
424		14.4	+ .39	48	2,1
131		14.43	+ .72	62	3,2
281		14.45	+ .38	80	4,4
332		14.45	+ .56	76	4,3
403		14.45	+ .22	67	2,2
142		14.47	+ .57	86	3,2
152		14.48	+ .67	162	6,4
176		14.48	+ .36	140	6,5
230		14.53	+ .78	90	3,3
401		14.53	+ .96	67	3,2
256		14.55	+ .75	142	4,4
191	8*s*	14.56	+ .66	148	5,4
13		14.6	+1.12	19	1,1
14		14.6	+1.08	38	1,1
327		14.6	+ .68	76	2,2
437		14.6	+1.10	19	1,1
439		14.6	+ .88	38	1,1
459		14.6	+1.30	19	1,1
480		14.6	+ .63	19	1,1
391		14.64	+ .52	64	5,3
265	9*s*	14.68	+ .86	128	4,4
154		14.7	+ .18	33	2,2
288	26	14.73	+ .85	66	3,3
60		14.8	+ .83	28	1,1
86		14.8	+ .38	28	2,2
105		14.8	+ .58	66	3,3
107		14.8	+1.36	28	2,2
111		14.8	+ .49	42	2,2

TABLE II—*Continued*

No.		m_{pg}	I_λ	$p(I_\lambda)$	n_m, n_λ
H.A. 48	N.P.S.				
125		14.8	+ .57	61	3,3
175		14.8	+ .31	42	2,2
333		14.8	+1.06	28	1,1
336		14.8	+1.05	28	2,2
338		14.8	+ .83	42	2,2
342		14.8	+ .19	28	2,2
348		14.8	− .41	14	1,1
226		14.85	+ .51	94	4,4
74		14.9	+ .90	28	2,2
324	27	14.97	+ .80	56	3,3
192		14.98	+ .37	56	4,4
88		15.0	+ .66	23	2,2
130		15.0	+ .65	51	3,3
140		15.0	+1.06	65	3,3
283		15.0	+1.28	23	2,2
311		15.0	+ .14	37	2,2
159		15.08	+ .43	65	4,4
158		15.12	+ .63	55	4,4
190	28	15.15	+ .75	50	4,4
334		15.15	+ .80	18	2,2
132		15.2	+1.25	18	1,1
306		15.2	+ .53	18	2,2
341		15.2	+ .91	27	2,2
323	10s	15.2	+ .56	36	3,3
196		15.30	+ .65	36	4,4
139		15.35	+1.72	13	2,2
193		15.35	− .04	17	2,2
277		15.35	+ .62	13	2,2
254	11s	15.38	+ .65	35	4,4
215		15.40	+1.50	44	4,4
126		15.43	+1.04	8	3,2
165		15.45	+ .43	13	2,2
163		15.5	+ .76	4	1,1
201		15.5	+ .19	11	2,2
205		15.5	+ .76	8	1,1
218		15.5	+1.12	12	2,2
358		15.5	+1.09	9	2,1
339	13s	15.6	+ .81	6	3,2
365	12s	15.65	+ .94	10	2,2
295		15.7	+ .28	4	2,1
164		15.90	+1.25	2	4,1
368		15.9	+1.28	2	1,1
212	29	15.95	+ .21	2	2,1
186		16.1	+ .57	2	1,1

mined. Of these the star 439, for which the photographic magnitude of Seares and consequently I_{MW} is uncertain, was omitted. Among the other 76 stars there are 40 for which the weight of I_λ is less than 90. These and the star 101 have been combined into means of 2 or more stars as follows: 420 and 101 (combined weight of I_λ 239); 195 and 282 (100), 319, 424, and 403 (141); 142 and 401 (153); 391 and 154 (97); 288 and 125 (127); 336, 338, and 342 (98); 324 and 130 (107); 283, 311, and 159 (125); 158 and 190 (105); 334, 341, 323, and 139 (94); 254, 215, 126, 163, 205, 218, 358, 339, 365, 295, 164, 368, 212, and 186 (148). Thus 35 single and 12 combined objects were available. These 47 objects are

TABLE III

m_m	I_{MW}	I_λ	I_λ O−C	m_m	I_{MW}	I_λ	I_λ O−C
10.08	+ .13	+ .16	+ 9	13.63	+ .60	+ .36	− 7
10.76	+ .44	+ .35	+ 2	12.90	+1.78	+1.50	0
10.42	+1.04	+ .93	+ 5	13.83	+1.04	+ .66	−17
10.82	+ .37	+ .22	− 5	13.44	+1.40	+1.07	− 8
10.58	+1.03	+ .90	+ 3	14.04	+ .96	+ .55	−20
11.06	+ .33	+ .17	− 6	13.83	+ .98	+ .88	+11
11.06	+ .65	+ .65	+13	13.98	+ .89	+ .41	−28
11.42	+ .39	+ .15	−13	13.84	+ .48	+ .30	− 3
11.42	+ .69	+ .66	+11	14.09	+ .82	+ .35	−28
12.08	+ .38	+ .21	− 5	14.16	+ .80	+ .67	+ 6
12.36	+ .51	+ .39	+ 2	14.12	+ .90	+ .74	+ 4
12.48	+ .41	+ .25	− 3	14.26	+ .81	+ .75	+14
12.76	+ .51	+ .23	−14	14.14	+ .71	+ .66	+13
12.72	+ .82	+ .54	−11	14.38	+ .72	+ .40	−13
12.64	+1.16	+1.11	+16	14.30	+ .90	+ .86	+17
12.67	+1.20	+ .96	− 3	14.45	+ .84	+ .72	+ 8
13.35	+ .50	+ .35	0	14.38	+ .89	+ .71	+ 3
13.12	+ .63	+ .44	− 3	14.58	+ .70	+ .51	0
12.90	+ .85	+ .79	+12	14.68	+ .81	+ .73	+12
13.20	+ .74	+ .60	+ 3	14.68	+ .81	+ .50	−11
13.15	+1.14	+1.01	+ 8	14.87	+ .62	+ .69	+26
13.04	+1.61	+1.21	−14	14.74	+1.00	+ .87	+ 9
13.58	+ .50	+ .32	− 3	14.86	+1.21	+1.02	+ 6
13.15	+1.38	+1.17	+ 3				

given in Table III, arranged according to the photographic magnitudes of Table II. The first column of Table III contains the mean of the photovisual and photographic magnitudes of Seares, m_m, which is independent of their difference I_{MW}. The material of Table III was treated according to least squares with the following result:

$$I_\lambda - 0.632 = \underset{\pm 0.054}{0.895}(I_{MW} - 0.810) - \underset{\pm 0.014\ \text{(m.e.)}}{0.015}(m_m - 13.128)$$

or written in another way

$$I_\lambda - 0.632 = 0.888(m_{P_g} - 13.533) - 0.902(m_{P_v} - 12.723).$$

The differences O−C between the observed and calculated values of I_λ are given in the last column of Table III. These differences do not show any distinct run, the number of changes in sign being 25, where we should expect 23±3.4 (m.e.). The linear formulae therefore represent the observations sufficiently well.

From the first of these formulae it is seen that there is no sensible magnitude equation between I_λ and I_{MW}, as the coefficient of m_m practically does not exceed its mean error. This result is perhaps the most satisfactory one of the present note. As described above, special care was taken to avoid a magnitude error in the values of I_λ. Although an error of this kind does not appear in the differences $m_{P_g} - m_{P_v}$ this does not prove that the scales for m_{P_g} and m_{P_v} are necessarily correct; but they must be parallel.

To say the least the values of I_λ do not indicate any such inequality between the two scales of Seares.

To $m_m = 13.128$ and $I_\lambda = 0$ corresponds $I_{MW} = +0.10 \pm 0.05$ (m.e.), although we have both intended to reduce our scale of color index to the same zero point. As the small disagreement found is only about twice its mean error its reality is not beyond question.

The mean square of the difference O−C in Table III is $0.0141 = (\pm 0.12)^2$. As this square is the sum of the squares of the mean errors of I_λ and I_{MW} the accuracy indicated is about what could be expected.

Considering the individual stars, it may be observed that the star 85, for which $I_\lambda = +0.30 \pm 0.08$ (m.e.), seems to be wrongly included with the *r* series (No. 10*r*), as it does not appear to be of deep color. My photographic magnitude 12.62 gives in connection with the photovisual magnitude 12.03 of Seares: $I_{MW} = +0.41$, corresponding to $I_\lambda = +0.29$, thus confirming the value found directly.

On September 14, 1912, a plate (No. 207) was taken with several exposures of the Pole star in order to determine the effective wave-length of its Σ companion 18″.3 distant. From six images was found $I_\lambda = +0.11$ with weight 188, or a mean error of ± 0.073. This value of I_λ corresponds to a spectrum of A0 to A5 of the Harvard classification. Recently F. C. Leonard (*Publ. Astr. Soc. of the Pacific*, **33**, 215, 1921) has found the spectrum of the companion to be of class F0.

Leiden
November 12, 1921

ELECTRIC FURNACE EXPERIMENTS INVOLVING IONIZATION PHENOMENA[1]

By ARTHUR S. KING

ABSTRACT

Furnace temperature required to obtain the subordinate series lines of the alkali metals in absorption.—With the central plug at about 2600°C, the stronger lines of both the first and second subordinate series of K, Cs, and Rb were readily obtained, while for Na, less than 2350° was required. The lines observed for each metal are listed. The temperature of the absorbing vapor may be taken as about 400° less than that of the plug.

Furnace absorption spectrum of iron vapor above 2800°C.—Using lower temperatures, no absorption lines had previously been obtained longer than λ 5507, but with a plug temperature above 3200°, practically all the high temperature emission lines were obtained in absorption, up to the end of the range photographed, λ 6700.

Furnace spectra of Ca, Sr, and Ba vapors mixed with more easily ionized elements, K and Cs.—*Flame lines* are not affected, but the *intensity of enhanced lines* was found to be decreased for both absorption and emission spectra, when the easily ionized vapor of K (4.3 volts) or of Cs (3.8 volts) was added, the effect being very marked for Ca (6.1 volts), somewhat less for Sr (reduction to one-third) and small for Ba (5.2 volts), while adding Na (5.2 volts) to Ba vapor produced no change. Evidently the *quenching effect* depends upon the relative ease of ionization of the two vapors. Temperatures of 1800–1830° were used.

Furnace temperatures required to obtain certain flame lines in absorption and in emission.—The lines λ 4227 (*calcium*), λ 4607 (*strontium*), and λ 5535 (*barium*) were obtained in absorption (tungsten lamp source) but not in emission (exposure 2 hours) at temperatures of 1275°, 1200°, and 1400°C respectively. This lag of emission over absorption may be partly due to the wideness of the lines, and is not noticeable for sharp *iron* lines.

Relation of furnace observations to Saha's theory of spectra.—The foregoing results verify Saha's predictions that the subordinate series of the alkalies could be obtained in absorption if sufficiently hot vapor were used, and that a gas too cool to emit light should still be able to absorb the lines of the principal series; and the results with mixed gases are such as would be expected if flame and enhanced lines are associated with neutral and ionized atoms respectively, as Saha assumes.

This paper gives the results of some experiments which bear on certain features of the theory recently advanced in a series of papers by M. N. Saha,[2] concerning the part which ionization may have in the production of certain types of spectral lines. The points tested are (1) the possibility of obtaining in absorption the lines of the subordinate series of the alkali metals, and in general, the conditions necessary for any type of lines to appear in absorption, (2) the effect, on lines presumed to be due to the ionized atom, of mixtures with elements having lower ionization potentials, and (3) whether, under given conditions of the metallic vapor, a line appears in absorption at a lower temperature than it can be obtained in emission.

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[1] *Contributions from the Mount Wilson Observatory*, No. 233.

[2] *Philosophical Magazine*, **40**, 472, 809, 1920; **41**, 267, 1921; *Proceedings of the Royal Society*, A, **99**, 135, 1921

The tube-resistance furnace offers special possibilities for an investigation of this sort by reason of the close control of the temperature and pressure. Absorption spectra also are readily obtained, either by the use of a diaphragm within the tube or by passing white light through the tube from an external source.

The special means of excitation employed will be described under each heading. The furnace chamber was usually pumped out to about 8 mm of mercury. The spectrograms were all made with the bright first order of a 15-foot concave grating, giving a scale of 3.7 A per mm.

I. ABSORPTION SPECTRA OF SODIUM, POTASSIUM, CAESIUM, AND RUBIDIUM

In his discussion of absorption spectra,[1] Saha notes that for these elements only the lines of the principal series have been obtained in absorption, the experiments with sodium having been by Wood[2] and those with potassium, caesium, and rubidium by Bevan.[3] Saha's explanation of the failure of the subordinate series to appear is that the temperature of the absorbing vapor was not high enough. In view of the writer's recent study[4] of the production in absorption of lines belonging to various temperature classes, this explanation seemed reasonable. The subordinate series of lines certainly require, in emission spectra, a stronger excitation than the lines of the principal series, and they should appear in absorption when the temperature is high enough to furnish a fairly large number of atoms capable of emitting the subordinate series lines. However, it was worth while to test this in the specific case of the alkali metals.

Sodium.—A trial of the furnace with sodium, operated as usual to give emission spectra, yielded at 2250° C and without prolonged exposure the pair λλ 5683–88 of the first subordinate series, and two pairs, λλ 5149–54 and 6155–61 of the second subordinate series, in addition to the widely reversed D lines. This showed that the production of the required atomic orbits was well within the range of the furnace.

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[1] *Philosophical Magazine*, **41**, 267, 1921.

[2] *Astrophysical Journal*, **29**, 97, 1909.

[3] *Proceedings of the Royal Society*, A, **83**, 421, 1910; **85**, 54, 1911.

[4] *Mt. Wilson Contr.*, No. 174; *Astrophysical Journal*, **51**, 13, 1920.

In order to obtain the absorption spectrum, a plug of graphite was placed near the middle of the furnace tube. The spectrograph was set to give, in addition to the pairs above mentioned, those at λλ 4979–83, 4748–52, 4665–68, the plate thus including three members each of the first and second subordinate series. A plug temperature of 2650°–2700° gave distinctly in absorption all six pairs. A temperature of 2350°, however, sufficed to show the three pairs of the first or diffuse series and the strongest pair of the second or sharp series at λλ 6155–61. The absence of the weaker sharp pairs was probably due to their extreme narrowness at low temperature in a vacuum source, so that the absorption lines were below the limit of photographic resolution. Aside from this feature, the sharp and diffuse series appeared in absorption with equal ease.

The temperature of the absorbing vapor was necessarily much below that of the plug, about 400° lower, according to the experiments previously reported,[1] which have since been confirmed so often that this value may be considered a constant of the special experimental arrangement employed. This means that near the end of the heated portion of the tube there is a layer of vapor sufficiently cooler than the plug to absorb radiation from the latter. The present experiments thus indicate that sodium vapor at 2000° C or higher, with a sufficiently hot background, will absorb the lines of the two subordinate series.

Potassium.—A trial with a plug temperature of 2700° gave in absorption the stronger lines of the subordinate series groups near λλ 5800 and 5340. These consist in each case of one pair belonging to the first and one to the second series.

Caesium.—The absorption spectrum with caesium chloride in the furnace was photographed from λ 5400 to λ 7000. All but the faintest of the arc lines in this region were obtained in absorption for a plug temperature of 2600°. The absorption spectrum showed nine lines of the first subordinate series and five weaker ones which Saunders,[1] by reason of the constant differences occurring among them, considers to belong to the second subordinate series. The two groups are as follows:

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[1] *Loc. cit.*

I		II
5466	6011	5746
5503	6213	5839
5635	6724	6034
5664	6974	6355
5845		6587

Three lines, given by Saunders as λλ 5847.8, 6217.6, 6983.8, and classed by him as satellites of adjacent strong lines, are visible also in the absorption spectrum. In the arc the lines which presumably belong to the second series are very unsymmetrical toward the red, while the absorption furnace in vacuum gives for each of these a narrow line at the extreme violet side of the position of the arc line. Dissymmetries occurring among the lines of the first series are also eliminated by the furnace. When the furnace spectrum is more completely photographed, with proper standards, a revision of the wave-lengths can be made and the character of the second series definitely settled.

Rubidium.—For this element also, there was no difficulty in obtaining in absorption nearly all of the lines given by the arc. In the range λ 5100 to λ 6400, the following lines of the first and second subordinate series showed as absorption lines, only the fainter members of the second series being absent:

I		II
5151	5648	5654
5196	5724	6071
5260	6207	6160
5363	6299	
5432		

While not bearing so directly upon the ionization theory as the foregoing, a series of experiments with the iron spectrum in absorption may be mentioned. In the previous investigation,[2] when the furnace was heated in vacuum with the plug at about 2600° C, no iron lines were obtained in absorption to the red of λ 5507, an effect probably due, as was explained then, to the high-temperature character of the iron lines in the yellow and red. Later experiments have extended this limit. The furnace, operated at atmospheric pressure, had its tube thinned in the central region to permit very high temperatures of the plug and to give a strong gradient in the portion of the tube containing the iron vapor. Plug temperatures above 3200° C, the highest reading of the pyrometer employed, were thus obtained. The spectrum was photographed to λ 6700, and gave in absorption practically all of the lines shown in emission by the furnace at high temperature.

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[1] *Astrophysical Journal*, **20**, 188, 1904.

[2] *Loc. cit.*

The conclusion from these results is that a sufficiently high temperature of the absorbing vapor, and proportionately higher temperature of the incandescent background, give in absorption all lines which the furnace can emit, and experience has shown these to include probably all lines not requiring conditions similar to those of the spark discharge.

This condition, which is an obvious consequence of the difference in excitation required to produce different groups of lines in the same spectrum, is enunciated by Saha to the effect that in order to reverse a given set of lines, it is not sufficient to send a beam of white light through a cooler layer of the vapor, but that "in the atoms present, there should be a fairly large number with orbits corresponding to the first term of the pulse of radiation to be absorbed. Thus in order that lines $(2p-md)$ may be absorbed (as those of the subordinate series of the alkalis), we must have a sufficient number of atoms with $(2p)$ orbits."[1]

2. EFFECT OF MIXTURES OF ELEMENTS HAVING DIFFERENT IONIZATION POTENTIALS

In the writer's studies of electric furnace spectra, it has been noted repeatedly that a mixture of substances may be vaporized without any suppression of the spectrum of one by reason of the presence of the others. In contrast with the arc containing such a mixture, each vapor appears to give its spectrum as if it were alone.

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[1] *Op. cit.*, p. 277

No evidence against this view has been found as regards the great majority of the lines emitted by the furnace. These lines, according to the modern view, are emitted by various electron orbits of the "normal atom," that is, an atom which is not ionized through the removal of one or more electrons. The furnace is capable, however, of emitting, in general rather faintly, some of the enhanced lines, which are regarded as due to the ionized atom. For these lines, the degree of ionization is a determining factor, and this may be affected by conditions other than the temperature and pressure which are controlled in the furnace.

The calcium spectrum contains various series emitted by the normal atom, and also the enhanced lines H and K arising from the ionized atom. Designating the normal atom by Ca and the ionized atom and its lost electron by Ca^+ and e, respectively, Saha[1] states, "At a given temperature and concentration, a definite equilibrium will be established between the proportions of Ca, Ca^+, and e, as represented by the van't Hoff formula of reversible chemical action, $Ca \rightarrow Ca^+ + e$."

If then we add a large supply of electrons from some substance which becomes ionized more readily than calcium, fewer electrons from the calcium can exist in the free state, and the proportion of Ca^+ atoms becomes less, with a corresponding weakening of the H and K lines, relative to the lines of the normal atom. Since the ionizing potential of an element may be taken as the measure of its ionization under a given stimulus, a mixture of calcium with an element having a lower ionization, such as potassium, caesium, or rubidium, should alter the calcium ionization to an appreciable extent. A fairly crucial test of the basic assumptions of Saha's theory should thus be possible, and Dr. H. N. Russell has suggested to the writer that the control of conditions in the electric furnace affords an excellent means of applying the test. The following experiments were accordingly carried out for the spectra of calcium, strontium, and barium.

Calcium.—A temperature of about 1800° C was selected, which gives the H and K lines distinctly, and yet with intensities low enough to register easily any disturbance in the vapor producing them. The first experiments were with absorption spectra, which permit short exposures, during which there would be little change in the vapor within the tube. Instead of using a plugged tube, white light from a strong tungsten projection lamp was sent through the tube of the vacuum furnace. As the filament of the lamp was many hundred degrees hotter than the furnace tube, it could safely be assumed that the vapor in the highly heated portion produced the absorption, instead of that near the end as was evidently the case when a plug was used. The first mixture tried was calcium and potassium, whose ionization potentials are 6.08 and 4.32 volts, respectively. The absorption spectrum was first photographed with metallic calcium alone in the tube. The furnace was then recharged with a mixture of calcium and potassium chloride, and a second exposure made under the same conditions as the first. The quenching effect on the H and K lines was very distinct. With calcium alone, H and K appeared as absorption lines of moderate strength, comparable with some of the lines in the group of six near λ 4300 and in the series triplet λλ 4425–55. When calcium and potassium were vaporized together, scarcely a trace of H and K was to be seen, while the comparison lines due to the normal atom were practically unchanged. The nearly equal width of the strong flame line λ 4227 in the two exposures indicated that the density of calcium vapor was nearly the same in both cases. The experiment was repeated, the tube being wiped out before charging with the calcium alone, but the potassium was not entirely removed, as was shown by the presence of the violet potassium pair. H and K were thus somewhat weakened in the first exposure, but much more affected in the second with a large supply of potassium.

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[1] *Proceedings of the Royal Society*, A, **99**, 139, 1921.

The comparison was repeated with a new tube and metallic potassium, instead of the chloride, for the second exposure. This eliminated a band structure in the H and K region which may have had some effect in the previous experiment. H and K were again much weakened with reference to the other calcium lines, though the narrower potassium lines showed that the metal did not give as effective a mass of vapor in the tube as was obtained with the chloride.

Experiments with emission spectra were next made, first with potassium chloride and then with caesium chloride, which were compared with calcium alone. The exposures were much longer, with resulting variations in the supply of vapor during the runs, but the effects observed for absorption were repeated, H and K being distinctly reduced in intensity when a mixture was used. As was to be expected, the effect of caesium, with an ionization potential of 3.81, appeared greater than that of potassium, but the effective quantities of the two may have been different, and at present it is desired to lay stress only on the general effect of a substance with lower ionizing potential.

Strontium.—The experiments with strontium were made first for the absorption spectrum with the chloride in the tube, at a temperature of 1860° C, then with a mixture of strontium chloride and potassium chloride. The enhanced lines λλ 4078 and 4216 were compared with the flame line λ 4607 and a group of strong furnace lines from λ 4722 to λ 4876, presumably due to the normal atom. As with calcium, the enhanced lines showed a decided effect from the mixture of potassium. λ 4078 was disturbed by a band spectrum, but λ 4216 was favorable for comparison, and relatively to the other lines, showed a reduction to not more than one-third of its former intensity. A repetition of the experiment, this time for the emission spectrum at 1830°, permitted both λ 4078 and λ 4216 to be observed. They were found to be weakened by the addition of potassium to approximately the same degree as in absorption. This tube was afterward used for the experiment with calcium alone and calcium with caesium in emission, mentioned in the preceding section. The stronger strontium lines persisted and the enhanced pair was much reduced by the addition of caesium.

Barium.—A series of experiments was made for the absorption spectrum of barium at λ 1830° C, using both barium chloride alone and also mixtures of either potassium chloride or caesium chloride. The enhanced lines λλ 4554 and 4934 were compared with λ 5535 and the strong furnace lines from λ 5778 to λ 6063. The effect on the enhanced lines was much smaller than with the corresponding lines of calcium and strontium, but a perceptible relative weakening of the enhanced pair followed the addition of both of the substances with lower ionizing potentials.

The ionization potential of barium is lower than that of strontium or calcium, being in fact halfway between potassium and calcium. This and the fact that the region occupied by the barium lines is not so favorable for the observation of absorption effects as the portion farther to the violet may be responsible in some measure for the smaller influence in the case of barium. A mixture with sodium vapor, whose ionization potential is almost exactly the same as that of barium, was tried and no effect whatever could be detected.

3. TEMPERATURES REQUIRED FOR THE INITIAL APPEARANCE OF A GIVEN LINE IN ABSORPTION AND IN EMISSION

It is desirable to extend the observations on this point much farther than has yet been possible, but the tests made on lines of a favorable type have given definite results, which will be reported here.

In his discussion of the radiation of the characteristic lines of the neutral atom,[1] Saha reasons that a mass of gas too cool to emit any light should still be able to absorb lines of the principal series. Further heating causes the atoms to progress toward ionization, and radiation will follow as a result of the mutual interchange of orbits. When the gas can emit the principal series rather strongly, it should absorb the lines of the subordinate series, a still higher temperature producing the latter series in emission.

The tests thus far made have been concerned with the question whether there is a temperature at which the principal series Ca λ 4227, Sr λ 4607, and Ba λ 5535 will appear in absorption and not in emission. The higher critical stage at which these lines are emitted and at which the subordinate series may show only in absorption has not yet been looked for.

The experimental arrangement provided a transition from absorption to emission conditions by simply turning off a 900-watt tungsten lamp, whose light was directed through the tube during the absorption exposure.

The exposures required for the absorption and the emission spectrograms were very different. For absorption one-half to one minute was sufficient to produce a continuous ground of the proper strength to show any absorption which was present. For the emission spectrum, it was a question of registering a very faint image, and a period of two hours was considered a fair time, in view of the power of the spectrograph, to show anything which the vapor could emit. Absorption spectra were photographed at the beginning and end of the emission run.

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[1] *Proceedings of the Royal Society*, A, **99**, 140, 1921.

At 1275° C, λ 4227 of calcium appeared in absorption, 0.5 A in width, persisting as a well-defined absorption line at the close of a two-hour exposure for the emission spectrum, but the line failed to show in emission. At the same temperature, a very faint trace of λ 4607 of strontium could be seen as an emission line, but nothing whatever appeared in emission at 1200°, though at the latter temperature the absorption line λ 4607 was strong at the beginning and end of the period.

The barium line λ 5535 could be seen very faintly in emission at 1500°; but with the temperature held close to 1400°, at no time rising above 1440°, the line entirely disappeared. The absorption line was still strong at 1400°.

Some inherent difficulties must be recognized in tests of this sort. Lines such as the three just considered have, when plenty of the material is present, low photographic density and a considerable width, which features should make them appear in absorption under minimum excitation. The same characteristics retard their development through the threshold condition in emission. Lines which tend, when faint, to become very narrow, as is the case with the great bulk of furnace lines, hold up relatively better in emission than in absorption as the temperature becomes lower; since in absorption the very narrow line may disappear too soon by reason of failure of the spectrograph to produce the resolution of the continuous spectrum which enables an absorption line to be perceived. This fact, which with the narrow lines of the vacuum furnace would make the emission line in its early stages easier to see than the absorption line, may account for the fact that in the iron spectrum, emission and absorption spectra for the same temperature of the furnace are very nearly duplicates. While the series for iron are not known, the lines fall into definite temperature classes,[1] and the initial appearance of a line of certain type may be watched. An iron line in this sensitive state and with proportional exposures, shows as readily in emission at a given temperature as in absorption. This was tested through a range extending from the ultra-violet into the red. From the green into the red, it is somewhat easier, at a certain temperature, to be sure of the presence of a faint iron line in emission than in absorption. We may therefore say that the evidence thus far does not always favor the absorption method as a means of producing more numerous lines than are shown under equal excitation in emission, though the earlier appearance in absorption of principal series lines appears to take place in the spectra of calcium, strontium, and barium.

MOUNT WILSON OBSERVATORY
January 1922

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[1] *Mt. Wilson Contr.*, No. 66; *Astrophysical Journal*, **37**, 239, 1913.

CARBON DIOXIDE ABSORPTION IN THE NEAR INFRA-RED

By E. F. BARKER

ABSTRACT

Infra-red absorption bands of CO_2 at 2.7 and 4.3 μ.—New absorption curves have been obtained, using a special prism-grating double spectrometer of higher resolution (Figs. 1–3). The 2.7 μ region, heretofore considered to be a doublet, proves to be a pair of *doublets*, with centers at approximately 2.694 μ and 2.767 μ. The 4.3 μ band appears as a single doublet with center at 4.253 μ. The frequency difference between maxima is nearly the same for each of the three doublets, and equal to 4.5×10^{11}. Complete resolution of the band series was not effected, even though the slit included only 12 A for the 2.7 μ region, but there is evidently a complicated structure, with a "head" in each case on the side of shorter wave-lengths. The existence of this head for the 4.3 μ band is also indicated by a *comparison with the emission spectrum* from a bunsen flame, and the difference in wave-length of the maxima of emission and absorption is explained as a temperature effect similar to that observed with other doublets.

Molecular structure of CO_2.—If a linear configuration of the three atoms is assumed, the *moment of inertia* of the molecule, computed from the doublet frequency difference, comes out 50×10^{-40}. This gives the distance from the central C atom to each O atom as 0.97×10^{-8} cm. Since the linear molecule could have only three modes of vibration, it is suggested that the two doublets at 2.7 μ may correspond to two successive changes in stationary state for the same vibration.

Echelette gratings, used in the research, were ruled with 3000, 1000, and 500 lines per inch, on surfaces of aluminum.

In the prismatic spectrum of carbon dioxide three infra-red absorption bands have been observed,[1] at approximately 2.7 μ, 4.3 μ, and 14.7 μ. The first region shows two absorption maxima with no indication of fine structure, but, from its behavior when the gas is subjected to various temperatures and pressures, von Bahr concluded that it is not a simple doublet. The region at 4.3 μ is narrow and for it no structure was observed. Using small echelette gratings, Trowbridge and Wood[2] found several maxima and minima in the emission spectrum from a bunsen flame between 4 μ and 4.5 μ, but they apparently did not take into account the reversal effect due to atmospheric absorption. These two regions have now been re-examined with the maximum available dispersion, using a prism-grating spectrometer designed for the purpose.

[1] Von Bahr, *D. Phys. Ges.*, **15**, 710 and 1150, 1913; Burmeister, *D. Phys. Ges.*, **15**, 589, 1913.

[2] *Philosophical Magazine*, **20**, 898, 1910.

The arrangement of the optical system is shown in Figure 1. The prism is of rock salt, with an angle of 38° and faces about 7×10 cm. It separates the incident beam into a spectrum which is focused at the plane of a slit forming the source for the grating spectrometer; thus a narrow spectral region is presented to the grating for analysis, and overlapping of spectra is avoided. The detecting system is a silver-bismuth thermopile actuating a high sensitivity galvanometer of the Paschen type. An absorption cell with windows of thin glass or mica is placed immediately behind the first slit, and may be moved into and out of the optical path.

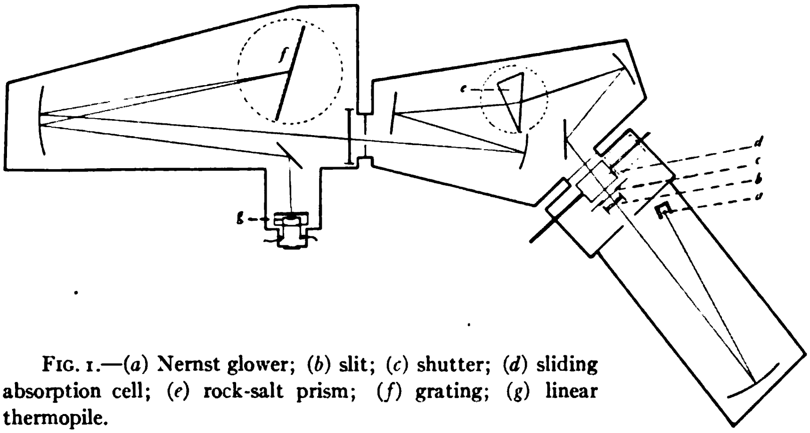

Fig. 1.—(*a*) Nernst glower; (*b*) slit; (*c*) shutter; (*d*) sliding absorption cell; (*e*) rock-salt prism; (*f*) grating; (*g*) linear thermopile.

The whole optical system is inclosed in a heavy metal case supplied with valves so that it may be thoroughly washed, dried, or even evacuated, as desired.

The region of 2.7 μ is accessible with a grating having approximately 15,000 lines per inch, and has also been examined in the first and second orders of a 5000-line grating. For the 4.3 μ band the latter grating failed to give spectra of measurable intensity and it has been necessary to prepare a number of coarser ones having approximately 3000, 1000, and 500 lines per inch. These have been ruled in the echelette form on 5″×7″ aluminium surfaces, and this department is indebted to the department of physics of Johns Hopkins University for the use of one of the Rowland ruling machines.

Observations.—Three absorption curves for the 2.7 μ band have been mapped with different dispersions, corresponding to approximately 55, 25, and 12 angstrom units, included by the half-millimeter slit. The absorption cell was five centimeters in length with glass windows about 0.25 mm thick. These extended beyond the cell so as to be traversed by the radiation even when the cell itself was removed from the optical path. Mica windows were used for greater wave-lengths, with a similar arrangement for compensation, but were avoided here on account of the absorption band of mica near 2.7 μ. The observations consisted in noting ten or twelve galvanometer deflections for a given setting of the grating, with the cell in and out alternately, from which the mean relative decrease in deflection due to the CO_2 absorption was computed. The interval between successive grating positions was somewhat less than the slit width.

The three different dispersions all show this absorption region to consist of a pair of doublets, so that we have, not a single typical band, but two, lying close together, with centers at approximately 2.69 μ and 2.76 μ. The use of different gratings, and the method of observing alternately with and without the absorption cell, completely eliminate the possibility of attributing this result to peculiarities of the ruled surface.

The curves of Figure 2 show the percentages of absorption obtained in the second-order spectrum of the 5000-line grating and the first-order spectrum of the 15,000-line grating. The existence of a complicated structure in these bands is very evident, but the available resolution is not sufficient to separate the individual members of the series. When using the 15,000-line grating the angle between the normal and the incident beam was 54°, so that obviously little is to be hoped for from finer gratings. The half-millimeter slit might be narrowed somewhat if a more intense source or a more sensitive detecting device were available.

The absorption band at 4.3 μ is narrow and intense, and may be observed as an atmospheric absorption even after long-continued washing of the spectrometer with CO_2—free air. That it is actually attributable to carbon dioxide is, however, established by the measurements with a 3 cm cell containing pure, dry CO_2, which absorbs more than 90 per cent of the incident energy throughout a considerable range of frequencies. With both the 1000-line and the 3000-line gratings the band appears as a distinct doublet, though there is little indication of the serrated structure which might be anticipated with higher dispersion. Figure 3A shows the atmospheric absorption obtained in the second-order spectrum from the 1000-line grating, the slit including 150 angstrom units. Figure 3B indicates the method employed to compute the percentage absorption, and Figure 3C is the absorption curve obtained in the first order of the 3000-line grating. The effect of introducing the absorption cell is shown in each. The dotted line of Figure 3B indicates the probable course of the energy curve for zero absorption, which could only be observed at the limits of the band. Its inclination to the horizontal is partly due to the falling off of energy as wave-lengths increase, and partly to the chosen prism setting.

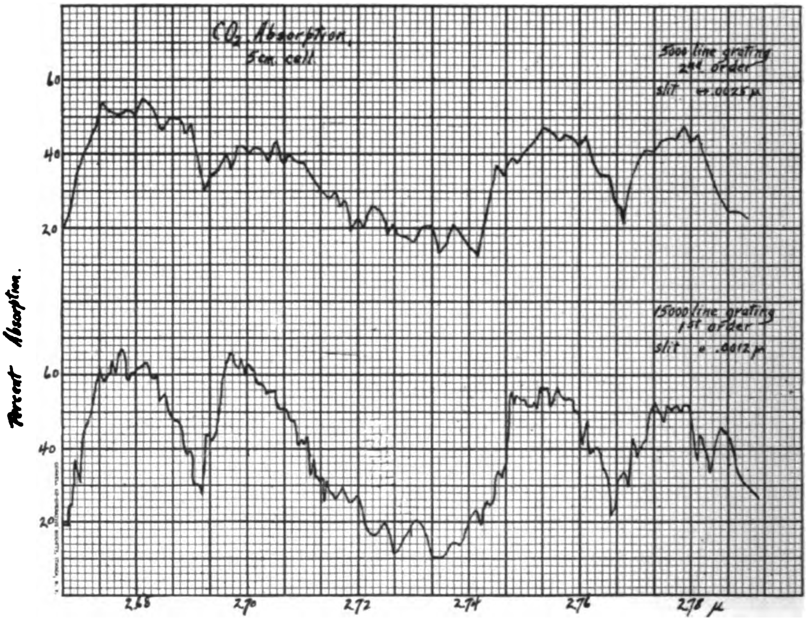

FIG. 2

The correction for slit width and slope of the energy curve[1] very slightly accentuates the doublet form, but does not displace the center nor the maxima appreciably.

The great intensity of the atmospheric absorption at 4.3 μ would seem to indicate that the emission band from a bunsen flame should be rapidly absorbed, though experimentally energy of approximately this wave-length is easily obtained through thick layers of air. The explanation is immediately obvious from Figure 4, which shows on the same scale the absorption of cold atmospheric CO_2 and the emission from the hot flame. As has been observed for other doublets, the widening due to increased temperature occurs almost entirely on the side of longer wave-lengths. From these curves it is evident that the doublet form would not be expected in the emission band unless precautions are taken to effectively eliminate absorption by cooler layers of CO_2—except, of course,

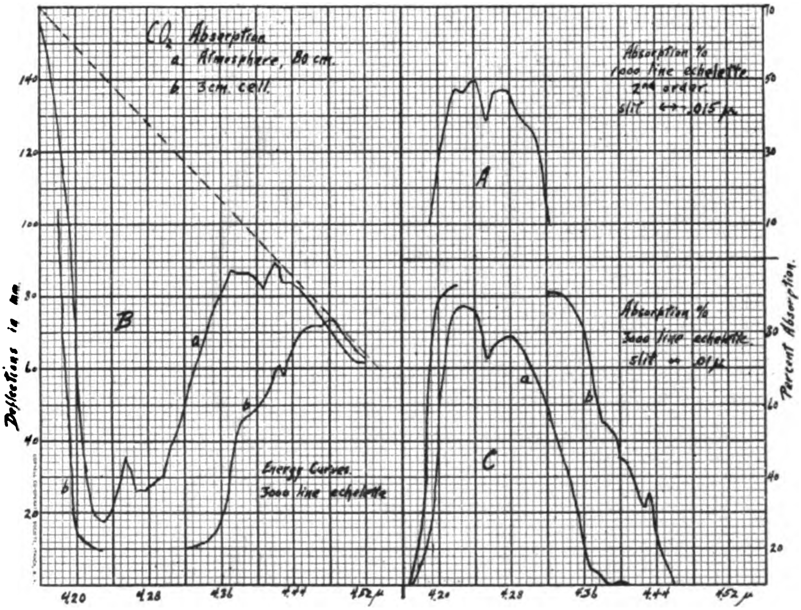

FIG. 3

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[1] Paschen, *Annalen der Physik*, 60, 712, 1897.

that the "reversal" reveals an inverted doublet produced by absorption, leaving three apparent maxima in the energy curve as observed. Of these, the weakest, corresponding to the central notch in the absorption curve, occurs at 4.25 μ, while the positions of the others, which we observe at approximately 4.2 μ and 4.4 μ,

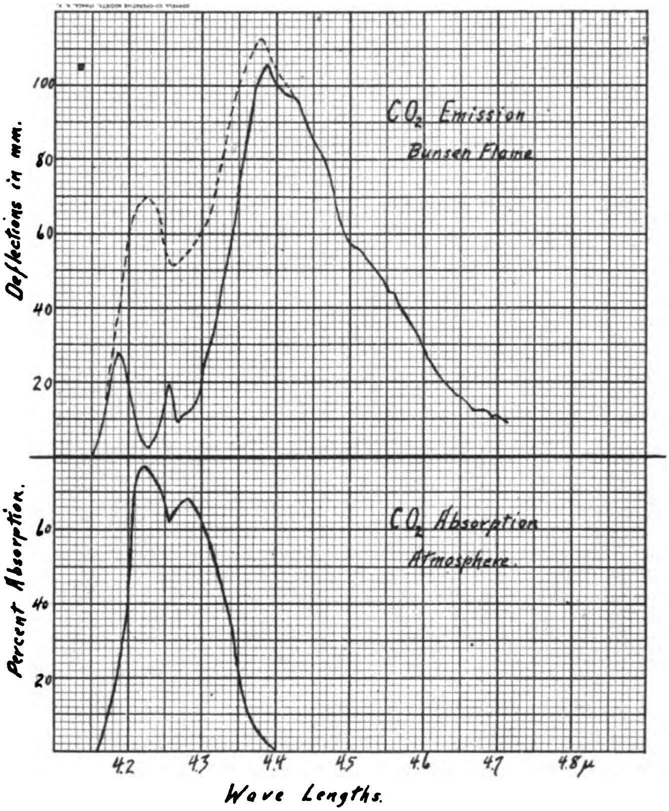

FIG. 4

depend upon the mass of gas in the cooler absorbing layers between the flame and the thermopile. Trowbridge and Wood (*loc. cit.*) observed maxima at 4.2 μ, 4.4 μ, and 4.5 μ with one grating, and at 4.32 μ and 4.43 μ with another. The dotted line in Figure 4 shows approximately what the energy curve would have been had no absorption occurred.

Although from the measurements here presented it is not possible to isolate the various band series, several interesting conclusions may nevertheless be drawn. In the doublet of Figure 3 and both those of Figure 2 the side toward shorter wave-lengths is steep while the bands appear broadened toward longer wave-lengths; also the higher frequency maximum is the more intense. This suggests band series similar to those of the halogen acids[1] in which the individual members are crowded on the side of shorter wave-lengths and separated by increasing intervals on the other side. For the 4.3 μ band it would appear that our measurements approach very close to the head of the series, since increases in the mass of the absorbing gas and in the temperature (Figures 3C and 4) build up the absorption almost entirely on one side.

A simple band series like that of HCl would be expected only if the CO_2 molecule were linear, as in Langmuir's model, with a single effective moment of inertia. The spacing of members near the middle of the band would then be about $h/4\pi^2 I$. A fair approximation to the value of I may be obtained from the frequency differences of the two maxima in the various doublets. Calling this interval $2\nu_m$, we have, since ν_m is the most probable frequency of rotation,

$$I=\frac{RT}{4\pi^2 N\nu_m^2}=\frac{1.11\times 10^{-15}}{\nu_m^2}.$$

Table I shows the values of ν_m taken from Figures 2 and 3, with corresponding values of I.

TABLE I

Band Center	Maxima	$\Delta\lambda$	ν_m	I
2.694	2.677, 2.698	.021	4.35×10^{11}	53.4×10^{-40}
2.767	2.753, 2.776	.023	4.52	49.5
4.253	4.225, 4.280	.055	4.56	48.6
Mean				50.5

For a symmetrical linear molecule with moment of inertia equal to 50.5×10^{-40}, each O atom would lie at a distance of 0.975×10^{-8} cm from the central C atom, a not improbable dimension.

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[1] Imes, *Astrophysical Journal*, **50**, 251, 1919, and Colby and Meyer, *Astrophysical Journal*, **53**, 300, 1921.

This value of I is, however, much larger than that derived from the thermal constants of the gas, assuming three principal moments of inertia, A, B, and C. Thus Schames[1] computes for carbon dioxide

$$\sqrt[3]{ABC}=5.80\times 10^{-40}.$$

Bjerrum[2] has discussed the triangular CO_2 molecule, deriving three possible vibration frequencies. From the laws of motion for the linear form it is easily shown that there are two degrees of freedom for vibration along the axis, to which we may add one vibration frequency for motion of the atoms at right angle to the axis. Thus either assumption apparently predicts three fundamental vibration spectra. Our analysis, however, reveals four bands, none of which may be considered as a so-called harmonic. The close proximity of the two bands near 2.7 μ suggests as a possible explanation that these may both be "fundamentals" of the same vibration, one corresponding to the vibrational transition from state 0 to state 1, the other to the transition from state 1 to state 2. Such a double fundamental would be expected if the motion is not simple harmonic, and has been predicted for HCl at 3.4 μ with a frequency difference for the two bands about the same as observed here.[3] This explanation would, however, involve the probable existence of a harmonic which has never been observed at about 1.4 μ, a region in which water vapor has an intense absorption band. The almost equal intensities of the two series of Figure 2 would indicate a number of molecules in the first state of vibration almost equal to the number in the zero state at ordinary temperatures. The ultimate decision as to whether or not the arrangement of atoms in the carbon dioxide molecule is linear or triangular must await an analysis of the bands with sufficiently high resolution to reveal the series of which they are constituted.

This investigation has been carried on at the Physical Laboratory of the University of Michigan under a Research Fellowship of the National Research Council.

UNIVERSITY OF MICHIGAN
January 1922

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[1] *Physikalische Zeitschrift*, **21**, 38, 1920.

[2] *D. Phys. Ges.*, **16**, 737, 1914.

[3] Kratzer, *Zeitschrift für Physik*, **3**, 289, 1920.

THE BROADENING OF THE BALMER LINES OF HYDROGEN WITH PRESSURE

By E. O. HULBURT

ABSTRACT

Photographic width of the Balmer lines of hydrogen for pressures up to 135 mm.—There is some question as to whether among the possible causes of the widening of lines with pressure, the decrease of mean free path is important. A theoretical calculation of the width to be expected for hydrogen lines if the length of the train of waves emitted is limited by the time between molecular collisions, gives for a temperature of 600° C. and a pressure of 100 mm a width of about 0.04 A, between wave-lengths where the intensity is one-tenth of that at the middle. Experimentally, however, it was found that a pressure of about 100 mm gave photographic images 80 A wide. Purified hydrogen in a capillary discharge tube was excited by condensed discharges and a series of spectrograms were taken for pressures from 0.2 to 135 mm. The exposure time was 30 minutes in each case and the discharge was regulated so that the intensity of the central portion of $H\beta$ was practically the same for all pressures. As shown by Figure 2, $H\beta$, $H\gamma$, and $H\delta$ behave very much alike. The fact that the actual width is much greater than that computed theoretically indicates that the widening of spectrum lines is probably due to some other cause, perhaps electrical fields.

Introductory.—The former Lord Rayleigh has summarized the causes which interfere with the absolute homogeneity of spectrum lines under five headings:[1] (1) The translatory motion of the radiating particles in the line of sight operating in accordance with Doppler's principle. (2) A possible effect of the rotation of the particles. (3) Disturbance depending on collision with other particles either of the same or of another kind. (4) Gradual dying down of the luminous vibrations as energy is radiated away. (5) Complications arising from the multiplicity of sources in the line of sight. To which is added a sixth cause proposed and investigated by Merton.[2] (6) The effect of an electric field on the radiating particle. The electric field may be that imposed on the electrodes of the tube or may be the field of neighboring atoms as suggested by Stark,[3] or a superposition of the two.

The researches of Fabry and Buisson[4] and Michelson[5] have demonstrated that for pressures of the luminous gas below one millimeter of mercury (1) is probably the only cause of broadening of the spectrum line. This cause is, however, impotent to explain the tremendously greater broadening produced in certain lines by the use of either greater pressures or condensed discharges, and therefore recourse must be had to the other causes mentioned. Of these the effects of (2) and (4) must perforce be omitted from discussion, for our present knowledge of atomic constitution and the nature of radiation is insufficient for a successful consideration of the effect of rotation of the luminous particle and of the damping of the vibration as the energy is radiated away. We also neglect (5), for this merely aggravates the effects of the other causes. There remain (3) and (6). From their recent work on the energy distribution in the broadened lines of hydrogen, helium, and lithium by high pressures stimulated by condensed discharges, Merton and Nicholson[1] concluded that cause (6) was all important in producing the broadening and that (2) played no part whatsoever. Their observations were concerned with the *form* of the broadened line and did not touch upon the *magnitude* of the broadening. In view of the novelty of this conclusion it seemed of interest to determine the magnitude of the broadening. The present work was carried out to this end and the photographic widths of Hβ, Hγ, and Hδ of the Balmer series of hydrogen were measured for pressures from 0.2 to 135 mm of mercury. The broadening which might be expected from cause (3) under the conditions of the present experiment was then calculated and was found to be considerably less than the observed broadening. This evidence thus supported the conclusions of Nicholson and Merton in that it threw doubt on the validity of (3) as a cause of the widening.

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[1] *Philosophical Magazine*, **29**, 274, 1915.

[2] *Proceedings of Royal Society*, **92**, 322, 1915.

[3] *Elektrische Spectralanalyse chemischer Atome*, 1914.

[4] *Journal de Physique*, **2**, 442, 1912.

[5] *Astrophysical Journal*, **2**, 251, 1895.

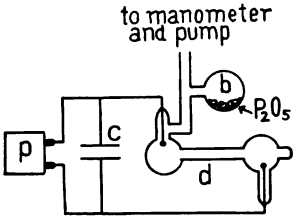

FIG. 1

Experimental.—Hydrogen, prepared from hydrochloric acid and zinc and purified through sodium hydroxide solution and concentrated sulphuric acid, was passed into the end-on discharge tube *d*, Figure 1, made of glass with aluminum electrodes arranged out of line with the

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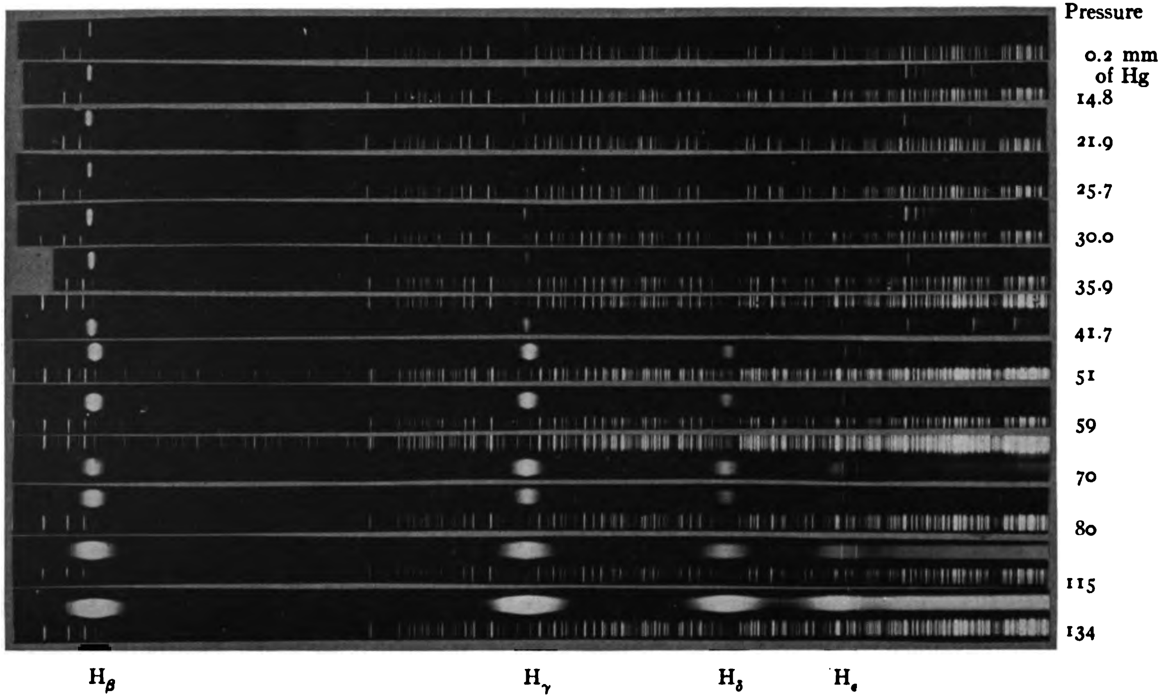
[1] *Philosophical Transactions*, Vol. 216, 459, 1916.

PLATE VII


Pressure
0.2 mm of Hg
14.8
21.9
25.7
30.0
35.9
41.7
51
59
70
80
115
134
H_β
H_γ
H_δ
H_ϵ

capillary. The capillary was 10 cm long and 5 mm in diameter. A bulb *b* with phosphorous pentoxide was sealed to the tube. The tube was connected by glass tubing about a meter in length to a bulb of perhaps 200 c.c. capacity and a mercury manometer. The total capacity of the discharge tube and glass system, perhaps 300 c.c., was filled with hydrogen at various pressures and no change in pressure was observed when the discharge was run continuously for an hour. The condenser *c*, Figure 1, of capacity of about 0.01 μ F, which discharged through the tube, was charged by the 25,000 volt transformer *p*. The spectra were photographed in the first order of a concave grating of 2 meters radius of curvature and ruled 14,435 lines to the inch, the dispersion being about 8.8 A per mm.

Spectra of hydrogen were taken for a series of pressures from 0.2 to 135 mm. These are shown in Plate VII with the iron arc comparison spectrum. The tube was maintained rigidly in position throughout the series and the time of exposure was 30 minutes for each spectrogram. In order to obtain conditions for the various pressures that might be compared the procedure was as follows: A glass prism spectroscope was trained on the hydrogen tube so that Hβ could be observed visually at the same time that the spectrum was being photographed. A small region at the center of the line was isolated by the second slit of the prism-spectroscope and observed by a low-power eyepiece. A comparison line was reflected into the field of view of the eyepiece obtained by illuminating a portion of the first slit with the light from a tungsten lamp filtered through a greenish-blue filter. The intensity of the comparison line was maintained constant, and by adjusting the power consumed in the hydrogen tube the intensity of the central portion of Hβ was kept approximately constant for each pressure. It is worthy of remark that the condensed discharge through the hydrogen tube was at all times an oscillatory one, the frequency as measured by a wave-meter being 0.28×10^6 alternations a second. For this reason the electrical power consumed in the tube could not be determined in a simple manner by electrical methods.

The widths in angstrom units of Hβ, Hγ, and Hδ were measured from the photographs of Plate VII, and were plotted in Figure 2 as ordinates against pressures in mm of mercury as abscissae. The measurement of the width was made simply by judging the points on the edges of the lines on the plates where the blackening became slight and measuring between these points. This was manifestly inaccurate but was deemed sufficiently precise for the purpose in hand. The last two points on each of the curves were too high, for at these two highest pressures it was not possible to reduce the intensity of the center of Hβ to the intensity used at the lower pressures and still maintain sufficient voltage to render the gas luminous.

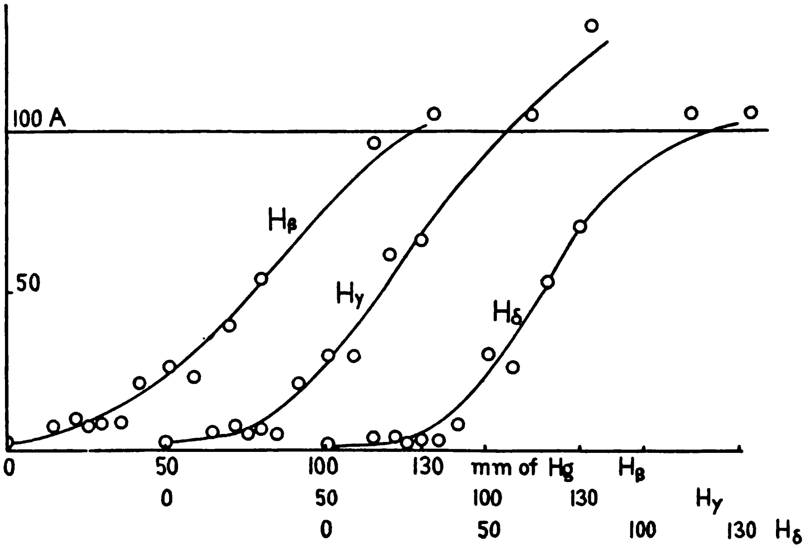


FIG. 2

Theoretical.—The third cause of widening mentioned in the introduction is the disturbance of the free vibration of the radiating particle due to encounters with other bodies. The simplest supposition open to us is that an entirely fresh start is made at each collision, so that we have to deal with a series of regular vibrations of finite length. The problem thus arising has been treated by Godfrey[1] and Schönrock.[2] The Fourier analysis of the limited train of waves of length r gives for the intensity of various parts of the spectrum line

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[1] *Philosophical Transactions*, **195**, 329, 1899.

[2] *Annalen der Physik*, **22**, 209, 1907.

$$I=I_0\frac{\sin^2\pi r\left(\frac{1}{\lambda_0}-\frac{1}{\lambda}\right)}{\left(\frac{1}{\lambda_0}-\frac{1}{\lambda}\right)^2} \tag{1}$$

where I_0 and λ_0 are the intensity and wave-length respectively of the center of the spectrum line and I and λ are the respective quantities in any part of the line. r is the length of the train of waves in cms. λ_0 is the wave-length which would be dominant if r were infinitely long.

We calculate r from the Kinetic Theory of Gases,[1] to be

$$r=\frac{c}{4\,\nu\sigma^2}\sqrt{\frac{m}{\pi \mathrm{RT}}} \tag{2}$$

where ν is the number of molecules per unit volume at the pressure and temperature in question, σ is the diameter of the sphere of molecular action from the Kinetic Theory standpoint, m is the mass of the molecule, c is the velocity of light *in vacuo* and R is the gas constant.

Substituting (2) in (1) gives

$$I=I_0\frac{\sin^2\frac{\pi c}{4\,\nu\sigma^2}\sqrt{\frac{m}{\pi \mathrm{RT}}}\left(\frac{1}{\lambda_0}-\frac{1}{\lambda}\right)}{\left(\frac{1}{\lambda_0}-\frac{1}{\lambda}\right)^2} \tag{3}$$

Formulae (2) and (3) assume slightly different forms depending on the methods used in averaging the velocities of the gas particles. These have been discussed by Tait,[2] Rayleigh,[3] and Godfrey,[4] and it seems that the various methods yield expressions which differ from (2) and (3) by not more than 10 per cent. This difference is small enough to be ignored in the present case.

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[1] Jeans, *Dynamical Theory of Gases*, p. 32.

[2] *Transactions Royal Society of Edinburgh*, **33**, 74, 1886.

[3] *Proceedings Royal Society, A*, **76**, 440, 1905.

[4] *Loc. cit.*

Comparison of theory and experiment.—By giving values to ν corresponding to the various pressures I was determined by means of (3) as a function of λ for each pressure, and from this the width of the spectrum line was calculated for each pressure. At a temperature 0° C. and 760 mm pressure ν was taken to be 4×10^{19}. The diameter of a hydrogen molecule was 2×10^{-8} cm. We assume that the gas was always at the temperature 600° C. $c=3\times10^{10}$, $R=9.3\times10^{-17}$, and $m=2.25\times10^{-24}$ grams. Introducing these values into (3) leads to

$$I=I_0\frac{\sin^2\frac{10634}{p}\left(\frac{1}{\lambda_0}-\frac{1}{\lambda}\right)}{\left(\frac{1}{\lambda_0}-\frac{1}{\lambda}\right)^2}, \qquad (4)$$

where p is the pressure of the hydrogen in mm of mercury. When I was computed from (5) for pressures within the range of the present measurements, curves of the type shown in Figure 3 were obtained. The width of the line increased with increase of pressure but was very narrow in all cases. For example, Figure 3 gives a curve for $\lambda_0=5000$ A and $p=100$ mm of mercury, and the width of the line represented by the curve is less than 0.04 A. In short, the broadening with pressure to be expected from the collision theory, or cause (3), is smaller by a different order of magnitude than that observed in the Balmer lines of hydrogen. In order to make the calculation agree approximately with the experimental measurements as far as the actual magnitude of the broadening is concerned, the number of equation (5) should be about 75 instead of 10634. The calculation was made on the basis

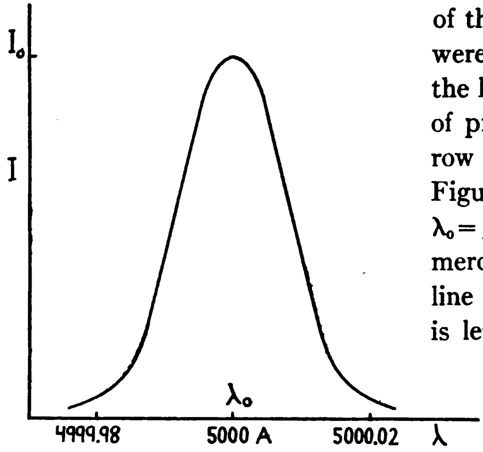

Fig. 3

of molecular hydrogen. Many researches, however, point to the fact that a disruptive discharge breaks the hydrogen molecule into its component atoms. If, then, we use values referring to the atoms, that is, employ a value for ν twice as great and for σ and m one-half as great as the molecular values previously substituted in (3), we find the number of equation (4) to be larger by the square root of two. This would make the calculated width of the line still smaller.

Discussion.—The discrepancy between theory and experiment is probably not to be found in experimental errors, approximate though the measurements were, or in a wrong assumption of the temperature of the gas. The discrepancy should rather be attributed to more fundamental errors of assumption, and in the light of the present experiment and the experiments of Merton and Nicholson (*loc. cit.*) we are ready to abandon cause (3) as inadequate and to turn to cause (6) as a possible explanation, as suggested by Merton (*loc. cit.*).

If the principal cause of the widening be attributed to the influence of an electric field on the radiating particle we may well inquire whether the electric field is the one imposed on the tube or is due to neighboring particles. Careful experiment on the separate influences of pressure and intensity of the discharge on the broadening combined with the known data of the Stark effect might differentiate between the two actions if they both exist. If the broadening is due to the electric field of neighboring particles we again return to the idea of collision. The effect of the collision upon the radiating particle, however, would be not merely a disturbance of the radiating centers by a mechanical shock but a profound momentary influence of an intense electric field.

STATE UNIVERSITY OF IOWA
IOWA CITY, IOWA
April 1922

AN ESTIMATE OF THE DISTANCE OF THE ANDROMEDA NEBULA

BY E. OEPIK

ABSTRACT

Andromeda Nebula.—Assuming the centripetal acceleration at a distance r from the center is equal to the gravitational acceleration due to the mass inside the sphere of radius r, an expression is derived for the *absolute distance* in terms of the linear speed v_0 at an angular distance ρ from the center, the apparent luminosity i, and E, the energy radiated per unit mass. From observations, v_0 comes out 157 km/sec. for $\rho = 150''$; and giving i a value corresponding to magnitude 6.1, and assuming E the same as for our Galaxy, the distance is computed to be 450,000 parsecs. This result is in agreement with that obtained by several independent methods. If it is correct, the *mass* within 150″ of the center is about 4.5×10^9 times the sun's mass, and the nebula is a stellar universe comparable with our Galaxy. *The ratio of the axes of the central ellipsoid*, whose shape is supposed to be due to rotation, was determined from photographs to be about 0.79.

Various estimates of the distance of the Andromeda Nebula have been made hitherto by H. Shapley,[1] H. D. Curtis,[2] K. Lundmark,[3] Luplau-Janssen and Haarh[4] and others; these estimates, based on the hypothesis that the Nebula consists of stellar matter similar to the matter of our Galaxy, lead to a distance of about 10^5 to 10^6 parsecs. Here we shall propose a method based on observed rotational movement, a method which may be applied to any nebula or cosmic system provided sufficient data are available. The principle is the same as that used by Campbell and Moore[5] to estimate the minimum mass of a planetary nebula.

Let v_0 be the velocity of motion along a circular orbit at a given point of the nebula, r, the distance from the center in astronomical units, M, the mass of the nebula within the radius $r(\odot = 1)$, w, the orbital velocity of the earth; then

$$\left(\frac{v_0}{w}\right)^2 = \frac{M}{r}. \qquad (1)$$

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[1] *Publications of the Astronomical Society of the Pacific*, **29**, 216, 1917.

[2] *Ibid.*, **29**, 206, 1917.

[3] *Astronomische Nachrichten*, **209**, 378, 1919.

[4] *Ibid.*, **215**, 285, 1922.

[5] *Lick Observatory Bulletin*, **9**, 4, 1916.

Let the integrated absolute luminosity $(\odot = 1)$ of the nebula within the radius r be I and the apparent luminosity $i(\odot = 1)$; then

$$\frac{i}{I} = \frac{1}{D^2}, \tag{2}$$

D being the distance in astronomical units. Let ρ be the angular distance from the center of the nebula, so that $r = D \sin \rho$, and let $E = \frac{I}{M}$; then, combining (1) and (2), we obtain

$$D = \frac{E \sin \rho}{i} \left(\frac{v_0}{w} \right)^2; \tag{3}$$

ρ, i, and w are known from observation; for v_0 an inferior limit is given by the relation $v_0 > \frac{v'}{\sqrt{2}}$, v' being the velocity of rotation or, more generally, of any internal movement, observed spectroscopically. Thus the estimation of the distance depends chiefly upon the quantity E, representing the average loss of energy per unit mass; this quantity plays an important part in Eddington's theory of the radiative equilibrium of stars.

If there exists absorption of light in space, and if ω is the fraction of light transmitted, and D' is the true distance, then the equations will be modified as follows:

$$\frac{i}{I} = \frac{\omega}{D'^2}; \tag{2'}$$

$$r = D' \sin \rho,$$

whence

$$D' = \frac{\omega E \sin \rho}{i} \left(\frac{v_0}{w} \right)^2 \tag{3'}$$

or

$$D' = \omega D, \tag{4}$$

D representing the distance computed without absorption.

Hitherto no assumptions as to the minute structure of the nebula have been made; if we postulate the existence of stars of a given absolute luminosity, and if α is their apparent luminosity computed without absorption, α', the same quantity corresponding to the absorption factor ω and distance D', then it is easy to obtain

$$\alpha' = \frac{\alpha}{\omega}. \tag{5}$$

Thus increased absorption diminishes the hypothetical distance but increases the apparent brightness of individual stars; this circumstance furnishes an upper limit of the hypothetical absorption, above which the individual stars of the nebula should become visible. Observational evidence indicates that if the Andromeda Nebula is a stellar universe with a luminosity-curve similar to that of our own stellar system, then ω cannot differ sensibly from unity.

For the Andromeda Nebula the following data may be used. F. G. Pease[1] has found a rotational velocity $v' = 72$ km/sec. at a distance of $\rho = 150''$ for which $\sin \rho = 0.000728$. To estimate the probable value of the circular velocity v_0, we shall suppose that the ellipsoidal shape of the inner parts of the nebula is a result of rotation. If a and b are the axes of the ellipsoid of revolution, $a > b$, $k = 1 - \frac{b}{a}$, then for small values of k and for a rotating non-homogeneous gaseous mass condensed at the center we have approximately

$$k = \left(\frac{v}{v_0}\right)^2, \tag{6}$$

v being the equatorial velocity of rotation. To obtain k, a plate taken with the 6-inch Petzval camera of the Dorpat Observatory, exposure 90 minutes, has been measured on a Hartmann microphotometer. Curves of equal density were constructed and the ratio of axes of these curves is given in the following table:

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[1] *Proceedings of the National Academy of Sciences*, **4**, 21–24, 1918.

Density of Negative	Millimeters		$b:a$
	$2a$	$2b$	
5.00	0.87	0.69	0.79
4.60	1.12	0.88	0.79
4.30	1.40	1.02	0.73

Scale of the photograph: 1 mm = 261″.

These data give for $a=150''$ the ratio $\frac{b}{a}=0.79$ and $k=0.21$. The angle between the line of sight and the equatorial plane of the nebula being small (about 8°) and the central condensation strong, we may assume that the values found represent the true ratio of axes of the ellipsoidal surfaces of equal density of the stellar or nebular matter within the nebula. Applying formula (6) with $k=0.21$ and $v=v'=72$, we obtain $v_0=157$ km/sec. This value must be regarded as an inferior limit, for in a stellar universe rotations in opposite directions may occur, and the spectroscopic velocity, representing the algebraic mean of all velocities, will be less than the mean absolute velocity.

As for E, we shall assume it equal to the corresponding value for our Galaxy. According to data given by Kapteyn and Van Rhijn,[1] 1000 cubic parsecs contain 44.7 luminous stars brighter than absolute magnitude 9.8, whose total luminous emission equals 55.5 stars of absolute magnitude zero[2] (nearly $=\odot$); J. H. Jeans[3] found that the observed motions of stars of our universe together with the distribution of density given by Kapteyn indicate that every luminous star counted by Kapteyn must correspond to an average mass equal to $3.2(\odot=1)$; thus we obtain $E=\frac{55.5}{3.2\times 44.7}=0.38(\odot=1)$ for our Galaxy.

The stellar magnitude of the Andromeda Nebula we shall assume $=5.0$; the stellar magnitude of the sun $=-26.6$ (corresponding to absolute magnitude zero). From a paper by Reynolds,[4] by a slight extrapolation, it has been found that the light of the nebula within 150″ from the center equals three-eighths of its total light; that corresponds to stellar magnitude 6.1. From these data we obtain $i=2.512^{-32.7}$.

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[1] *Mt. Wilson Contr.*, No. 188; *Astrophysical Journal*, **52**, 23, 1920.

[2] F. H. Seares, *Mt. Wilson Contr.*, No. 191; *Astrophysical Journal*, **52**, 167, 1920.

[3] *Monthly Notices*, **82**, 133, 1922.

[4] *Ibid.*, **74**, 132, 1914.

Substituting the data found above into equation (3), we find $D=9.10^{10}$ astron. units or 450,000 parsecs. From equation (1) we obtain $M=1.8\times 10^9(\odot=1)$ within 150″ distance from the center. For a stellar system the mass within a given volume may be assumed proportional to the total luminous emission; thus the total mass of the Andromeda Nebula must be eight-thirds times greater or 4.5×10^9 times our sun's mass. This would correspond to about 1500 million of stars brighter than 1/10,000 of our sun, if the luminosity-curve and distribution of mass in the nebula were the same as in the regions of space surrounding us.

The order of the distance found above is in agreement with other estimates of the distance of the Andromeda Nebula; the coincidence of results obtained by several independent methods increases the probability that this nebula is a stellar universe, comparable with our Galaxy. It may be added that in our computation this statement did not enter as an initial hypothesis. The only assumption concerning the constancy of the energy-production E may be true for any great aggregation of mass, without regard to the structure of the system; if we suppose that the energy of celestial bodies has its source in some processes like radioactivity, then our assumption would mean that the percentage of radioactive matter in the system of the Andromeda Nebula and in the Galactic system are equal.

TARTU (DORPAT)
April 7, 1922

NOTE ON THE THICKNESS OF AIR REQUIRED TO PRODUCE THE ATMOSPHERIC ABSORPTION BANDS, A, *a*, AND B[1]

By ARTHUR S. KING

ABSTRACT

Thickness of air required to give the absorption bands A, a, and B.—On a plate stained with dicyanin, the A and B bands were obtained when the light from a tungsten lamp traversed minimum thicknesses of only 7 and 40 meters respectively; and though the air was hot and dry, the water band *a* at λ 7200 was obtained with absorption of 9.5 meters. To give the bands A and B visually respectively twelve and five times greater thicknesses of air are required.

In the course of experiments on absorption spectra in which the light from a 900-watt tungsten lamp passed through a heated graphite tube and thence to a 15-foot concave grating spectrograph, it was noted that the A band at λ 7600 frequently appeared on the spectrograms. This band is ascribed to the absorption of oxygen in the atmosphere. A test with the furnace cold showed that the absorption was due to the air at room temperature. The distance traversed by the beam through the air in the laboratory and in the spectrograph was about 12 meters. As this air-path is much shorter than that usually regarded as necessary to produce this band, other distances were tried in order to find, if possible, the minimum thickness required to give A, and also under what conditions the companion band B near λ 6900 could be obtained.

Placing the lamp as near the slit as could be conveniently arranged gave an air-path of 9.5 meters. The A band was still fairly strong and evidently would appear with a path considerably shorter. A surprising feature was that the water-vapor band a at λ 7200 could be detected at this distance. It was just at the limit of visibility on spectrograms having a favorable density of the continuous ground. As a part of the air-path was in the spectrograph pit, a determination of the humidity was not attempted; but the experiments were made on a particularly dry day, a hot wind blowing from the Mojave desert, so that the water-content of the air, both in the laboratory and in the ventilated pit, must have been very low.

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[1] *Contributions from the Mount Wilson Observatory,* No. 232.

Changing the air-path to 19 meters strengthened both A and *a*, but B was still invisible. The light was then passed the length of the laboratory to a plane mirror and back to the spectrograph, a distance of 39.4 meters. B now appeared, extremely faint, but the structure of both branches of the band could nevertheless be made out. Both A and *a* were strongly developed.

To find how small a layer of air would give A, it was necessary to use a shorter-focus spectrograph, and a 1-meter concave grating was tried. This involved a greatly reduced scale and, as the component lines of the atmospheric bands produced in this way are very fine, the matter of photographic resolution became important. Finally, by using a fine-grained film stained with dicyanin, the A band was detected with an air-path of seven meters.

These minimum distances of seven meters for A and approximately forty meters for B are much shorter than those given by Liveing and Dewar,[1] who extended the earlier observations of Egoroff and Janssen. Their tests, however, were visual, and the red-sensitive plate is certainly far more effective for the observations of A; while in the B region the plate probably still has an advantage over the eye in the detection of faint objects, on account of the possibility of adjusting contrast and the strength of the continuous ground.

Liveing and Dewar found that 7 atmospheres pressure in a steel tube 18 meters long rendered A visible, while 18 atmospheres were required to show B. The same tube filled with pure oxygen at atmospheric pressure showed A, while two atmospheres of oxygen were needed for B to appear. From this observation, A should be discernible through 90 meters of air. Egoroff[2] observed it through 80 meters of air, or nearly twelve times the amount found necessary in the writer's experiments by the photographic method.

MT. WILSON OBSERVATORY
November 1921

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[1] *Proceedings of the Royal Society*, 46, 222, 1889.

[2] *Comptes Rendus*, 101, 1143, 1885.

SPECTROPHOTOMETRIC METHODS FOR DETERMINING STELLAR LUMINOSITY. A CORRECTION

Corrected parallax for Messier 11.—In the *Publications of the Astronomical Society of the Pacific*, **34**, 115, 1922, Dr. K. Lundmark has corrected an unfortunate error in my paper, *Mt. Wilson Contr.*, No. 228; *Astrophysical Journal*, **55**, 85, 1922. For the cluster Messier 11 I found the modulus $M-m=-12$, which corresponds to the parallax 0.″00040 and the distance 2500 parsecs or 8000 light-years (erroneously given 0.″00025, 4000 parsecs, and 13,000 light-years).

Bertil Lindblad

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